

Nonbilocal measurement via an entangled state

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(Received 2 January 2005; published 16 August 2005)

Two observers, who share a pair of particles in an entangled mixed state, can use it to perform some nonbilocal measurements over another bipartite system. In particular, one can construct a specific game played by the observers against a coordinator, in which they can score better than a pair of observers who only share a classical communication channel. The existence of such a game is an operational implication of an entanglement witness.

DOI: [10.1103/PhysRevA.72.022315](https://doi.org/10.1103/PhysRevA.72.022315)

PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

The relationship between entanglement and nonlocality of quantum systems has been the subject of extensive research. The most celebrated manifestation of the nonlocal aspect of entanglement is Bell's theorem [1], that correlations of outcomes of measurements over a pair of particles at singlet state cannot be squared with a local hidden variable model. This theorem was later extended to every pure entangled state [2]. The case of mixed states is more challenging. Werner ([3]) constructed an entangled bipartite state that admits a local hidden variables model that reproduces all the statistical correlations of von Neumann (ideal) measurements over the subsystems. In particular, the correlations of outcomes of local ideal measurements on a pair of particles at Werner's state do not violate any Bell inequality. But it was later discovered that Werner's states manifest some other nonlocal aspects [4–6]. The question therefore arises, does every entangled state manifest some aspect of nonlocality?

The concept of entanglement is easily generalized from pure states to mixed states. A non-negative operator F over a tensor product of Hilbert spaces is called *separable* if it can be written in the form $\sum_i K_i \otimes K'_i$, where K_i , and K'_i are non-negative operators. A mixed state of a bipartite quantum system is called *entangled* if the corresponding density operator is nonseparable. These are well-defined mathematical concepts, which are somehow related to the more vague physical concept of nonlocality. As mentioned above, several aspects of nonlocality have been suggested in the literature. The purpose of this paper is to present a facet of nonlocality that is manifested by *any* entangled state: Observers who share a pair of particles in that state can use it to perform nonbilocal measurements over another pair.

A measurement [or a positive operator-valued measure (POVM) measurement] is represented by a k -tuple of non-negative operators (F_1, \dots, F_k) such that $F_1 + \dots + F_k = I$. If the state of a system is represented by the density operator W and the POVM measurement (F_1, \dots, F_k) is performed over the system, the outcome is i with probability $\text{tr}(WF_i)$. Particularly important for this paper is the case $k=2$, i.e., measurements with two possible outcomes, “yes” or “no.” We

call such measurements *yes-no measurements*. A yes-no measurement is given by an operator F such that $0 \leq F \leq I$. If the state of a system is W and the yes-no measurement F is applied, the measurement's outcome is yes with probability $\text{tr}(WF)$, and no with probability $1 - \text{tr}(WF)$. Yes-no measurements are called effects in [7]. A POVM measurement (F_1, \dots, F_k) is called *local* if it can be carried out by Alice or Bob. This means that either $F_i = F_i^{(A)} \otimes I$ for each i [in which case, to perform the measurement Alice alone has to perform the measurement $(F_1^{(A)}, \dots, F_k^{(A)})$ on her particle], or that $F_i = I \otimes F_i^{(B)}$ for each i . A POVM measurement is called *bilocal* ([8]) if it can be performed by a sequence of local measurements and classical communication. Note ([8]) that the operator F corresponding to a yes-no bilocal measurement is necessarily separable.

In order to get some intuition about how mixed entangled states can be used to perform nonbilocal measurements, we first consider two examples. Assume that all particles have spin $\frac{1}{2}$ and that Alice and Bob share a pair of particles in a singlet state, which is given by the density operator $\rho_{\text{singlet}} = \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)$. If they are now introduced to another pair of particles at an unknown state W , they can use the singlet pair ρ_{singlet} to teleport [9] Alice's part of W to Bob. Bob now holds a pair of particles at state W , to which he can apply any yes-no measurement. Thus, using the singlet pair, Alice and Bob are able to perform nonbilocal measurements over the new pair. In particular, they can deduce more information about the unknown state W than can a pair of observers who can only communicate classically.

Consider another example. Suppose that Alice and Bob share a pair of particles at Werner's state, which is given by the density operator $\rho_W = \frac{1}{2}\rho_{\text{singlet}} + \frac{1}{8}I$, where I is the identity operator. Werner ([3]) showed that, even though this state is entangled, there exists a local hidden variables model that reproduces the correlations of all ideal local measurements that can be performed on it. Still, as was shown by Popescu ([4]), the nonlocal aspect of Werner's state is revealed when one tries to use it in the teleportation scheme instead of the singlet. This yields teleportation with better fidelity than the maximal fidelity that can be achieved by using only classical communication. We now show how this imperfect teleportation can be used to perform some nonbilocal measurement over a pair of particles at state W . Assume that Alice and Bob

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try to transfer Alice's part of W to Bob using the teleportation scheme with Werner's state ρ_W . Note that Werner's state can be seen as a mixture of the singlet ρ_{singlet} with a completely random state $\frac{1}{4}I$. Thus, with probability 0.5, the teleportation succeeds and Bob holds a pair of particles in state W . With probability 0.5, the teleportation fails, transferring the completely random spin- $\frac{1}{2}$ particle at state $\frac{1}{2}I$ to Bob. Thus, after this process Bob holds a pair of particles at state $\frac{1}{2}W + \frac{1}{4}I \otimes \text{tr}_A(W)$, where $\text{tr}_A(W)$ is the partial trace over subsystem A of W (which represents the state of Bob's part of W before the measurement). Suppose now that Bob performs the yes-no measurement given by the operator ρ_{singlet} on this pair. The probability to receive outcome yes is given by

$$\text{tr} \left[\left(\frac{1}{2}W + \frac{1}{4}I \otimes \text{tr}_A(W) \right) \rho_{\text{singlet}} \right] = \text{tr}(W\rho_W).$$

Thus using local measurements and classical teleportation, Alice and Bob simulated the yes-no measurement given by the operator ρ_W . Since ρ_W is nonseparable, this is a nonbilocal measurement.

Thus, the nonlocality of Werner's state is revealed by the fact that observers can use it to perform a nonbilocal measurement. In Sec. II we show that *every* entangled state ρ manifests this aspect of nonlocality: A pair of observers who share this state can use it to perform some nonbilocal yes-no measurement. In Sec. III the possibility of performing nonbilocal yes-no measurements using an entangled state is given a game-theoretic interpretation: We consider a game played by a pair of players, Alice and Bob, against a game coordinator, in which Alice and Bob have to guess the state of a bipartite system prepared by the coordinator. It is shown that if Alice and Bob share an entangled state they gain an advantageous guessing strategy by using the nonbilocal measurements.

It is interesting to compare the result of this paper with another aspect of nonlocality, namely distillation. An entangled state ρ is called *distillable*, if it is possible to create, with high probability, a singlet state from a large set of copies of ρ using only local operations and classical communication. It is known ([10,11]) that every pure entangled state is distillable, but there exist mixed states that cannot be distilled. These states are sometimes called *bound entangled states*. The fact that for every entangled state ρ there exists a state-guessing game in which sharing ρ is advantageous shows that even bound entangled states are still useful in certain situations.

The link between the nonbilocal measurement presented in Sec. II and the state-guessing game presented in Sec. III is an entanglement witness. Entanglement witness can be viewed geometrically as a hyperplane that separates an entangled state from the convex set of separable states. It is known ([12]) that every entangled state ρ admits an entanglement witness and ([13]) that the distance between ρ and the set of separable states in the Euclidian space of Hermitian operators equals the maximal violation of the corresponding "generalized Bell inequality" (see also [14] for the relationship between entanglement witnesses and distillation, and [15] for the use of certain entanglement witnesses to prove

the presence of entanglement in order to establish a secure key distribution.) The fact that every entanglement witness gives rise to a specific game, in which the players benefit from sharing ρ is an operational implication of the entanglement witness. Thus, this paper shows that the existence of an entanglement witness is not only necessary for a state to be entangled, but is also sufficient for the state to reveal nonlocality.

II. SCHEME FOR NONBILOCAL MEASUREMENT

In this section we describe a scheme for performing a nonbilocal measurement using a preprepared entangled pair ρ .

Let ρ be a nonseparable density matrix over $\mathcal{H}_A \otimes \mathcal{H}_B$. Consider a pair of particles at state ρ and assume that Alice has access to the particle that lives in \mathcal{H}_A and Bob has access to the particle that lives in \mathcal{H}_B . Assume now that Alice and Bob are introduced to another pair of particles at the unknown state represented by the density matrix W over $\mathcal{H}'_A \otimes \mathcal{H}'_B$, such that $\dim(\mathcal{H}'_A) = \dim(\mathcal{H}_A) = n$, and $\dim(\mathcal{H}'_B) = \dim(\mathcal{H}_B) = m$. Thus, the joint state of the four particles is represented by the density matrix $\rho \otimes W$ over $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}'_A \otimes \mathcal{H}'_B$. Alice has access to the subsystem $\mathcal{H}_A \otimes \mathcal{H}'_A$ and Bob has access to the subsystem $\mathcal{H}_B \otimes \mathcal{H}'_B$.

Let $\{|i\rangle\}, \{|i'\rangle\}, \{|\mu\rangle\}, \{|\mu'\rangle\}$ be orthogonal bases for $\mathcal{H}_A, \mathcal{H}'_A, \mathcal{H}_B, \mathcal{H}'_B$, respectively. Note that Latin indexes correspond to the particles held by Alice, and Greek indexes correspond to the particles held by Bob. Let $|\phi_A\rangle = (1/\sqrt{n})\sum_i |i\rangle \otimes |i'\rangle$ and $|\phi_B\rangle = (1/\sqrt{m})\sum_\mu |\mu\rangle \otimes |\mu'\rangle$. Assume that Alice and Bob perform the yes-no measurement $|\phi_A\rangle\langle\phi_A| \otimes |\phi_B\rangle\langle\phi_B|$ on the four-particle system $\mathcal{H}_A \otimes \mathcal{H}'_A \otimes \mathcal{H}_B \otimes \mathcal{H}'_B$. Note that this can be done by local measurements and classical communication: Alice performs the yes-no measurement $|\phi_A\rangle\langle\phi_A|$ over $\mathcal{H}_A \otimes \mathcal{H}'_A$, Bob performs the yes-no measurement $|\phi_B\rangle\langle\phi_B|$ over $\mathcal{H}_B \otimes \mathcal{H}'_B$, and the outcome of the measurement is given by the logical conjunction of the local outcomes received by Alice and Bob (thus, classical communication is needed to establish the outcome of the measurement from the outcome of the local measurements).

One can verify that, for every density matrix W over $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$\begin{aligned} & \text{tr}[(|\phi_A\rangle\langle\phi_A| \otimes |\phi_B\rangle\langle\phi_B|)(\rho \otimes W)] \\ &= \frac{1}{nm} \sum_{i,j,\mu,\nu} \langle i\mu|\rho|j\nu\rangle \langle i'\mu'|W|j'\nu'\rangle = \frac{1}{nm} \text{tr}(W\rho^t), \end{aligned}$$

where ρ^t is the transpose of ρ with respect to the basis $\{|\mu\rangle\}_{i,\mu}$ of $\mathcal{H}_A \otimes \mathcal{H}_B$. Thus, this scheme effectively performs the yes-no measurement $(1/nm)\rho^t$ over W . But since ρ is a nonseparable matrix, it follows that $(1/nm)\rho^t$ is also nonseparable. Thus using this scheme, Alice and Bob perform a nonseparable, and, in particular, a nonbilocal measurement over the state W .

III. A STATE-GUESSING GAME

In this section we try to shed some light on the implications of the nonbilocal measurement constructed above. To

do so, we describe a specific *game* that Alice and Bob play against a game coordinator, in which they can use the non-bilocal yes-no measurement $(1/nm)\rho^t$ to score better than a pair of observers who can only communicate classically. The discussion follows standard game-theoretic arguments.

Let H be an *entanglement witness* ([12]), i.e., an Hermitian operator such that $\text{tr}(H\rho) < 0$ but $\text{tr}(HD) \geq 0$ for every separable D . The existence of such an operator H follows from the inseparability of ρ and the separation theorem for convex cones ([16]). Let H^t be the transpose of H with respect to the basis $\{|i\mu\rangle\}_{i,\mu}$ of $\mathcal{H}_A \otimes \mathcal{H}_B$. We can assume that $H^t = \beta W^2 - \alpha W^1$ where W^1 and W^2 are density operators, and $\beta, \alpha \geq 0$. Since $\text{tr}(H^t) = \text{tr}(H) \geq 0$, it follows that $\beta \geq \alpha$.

Suppose that Alice and Bob are engaged in the following game: At the beginning of the game, a pair of particles is prepared by the game coordinator at state W^1 or W^2 with probabilities $\alpha/\alpha+\beta, \beta/\alpha+\beta$, respectively. The first particle is given to Alice and the second to Bob. Alice and Bob, who share a classical communication channel, know the parameters of the game (i.e., W^1, W^2, α, β) and their goal is to guess which state was actually chosen. They receive a payoff of +1 for a correct guess and -1 for an incorrect guess.

Every strategy that Alice and Bob can apply in the game corresponds to some yes-no measurement F on the pair of

particles: If the outcome of the measurement is yes they guess that the state was W^1 ; if the outcome is no, they guess that the state was W^2 . Their expected payoff is thus given by

$$\frac{\alpha}{\alpha+\beta} \{[\text{tr}(W^1 F)] - \text{tr}[W^1(I-F)]\} + \frac{\beta}{\alpha+\beta} \{[\text{tr}[W^2(I-F)] - \text{tr}(W^2 F)]\} = \frac{\beta-\alpha}{\alpha+\beta} - \frac{2}{\alpha+\beta} \text{tr}(H^t F).$$

If Alice and Bob can only perform local measurements and communicate classically, the yes-no measurement F they employ is necessarily a separable operator, and their expected payoff is therefore no greater than $(\beta-\alpha)/(\alpha+\beta)$. If, on the other hand, Alice and Bob share a bipartite system at state ρ , they can implement the scheme described in Sec. II and thus achieve a payoff $(\beta-\alpha)/(\alpha+\beta) - \{2/[nm(\alpha+\beta)]\} \text{tr}(H^t \rho^t)$. Since $\text{tr}(H^t \rho^t) = \text{tr}(H\rho) < 0$, this is strictly greater than the payoff they can achieve without this system.

ACKNOWLEDGMENTS

I am grateful to Ehud Lehrer for helpful discussions.

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