Introduction

Under its minimalist interpretation quantum theory is a set of rules for calculating the probability of outcomes of measurements over a physical system given previous information about the system [5]. Information about a physical system, of course, is not an inherent part of the system, but depends on the knowledge of an *observer*. Thus, according to this interpretation, quantum theory – indeed, every physical theory – does not venture to describe any *objective* element of reality, but to establish the mathematical framework for *subjective descriptions* of reality.

The purpose of this introductory section is two fold: to present the mathematical framework of quantum mechanics, and to formulate some well-known results from decision theory, namely Savage's subjective probability theory and Blackwell's theorem about comparison of experiments, in a way that illuminates their connection with classical physics. I do not aim to provide a comprehensive survey of quantum mechanics and its interpretations, but only to illuminate those aspects of the theory that seem relevant to decision theory, and especially those that appear in my papers.

Lattices of Physical Properties

Following von-Neumann [4], I shall use the term *physical property* to refer to something that can be said to be either *true* or *false* for a particular physical system. In this section I describe the mathematical framework of operations and relations between properties of a physical system. I work separately on classical and quantum systems. By a classical (resp. quantum) system I mean a physical system that behaves according to the laws of classical (resp. quantum) physics. I make no assertions about the scope of each theory, since my interest is not physics *per se*. For my purposes it is sufficient to consider classical and quantum systems as two different and legitimate mathematical models of physical systems.

To every classical physical system there corresponds a *phase space*, which in the case of a system with N particles, is a subset $\mathcal{S}$ of $\mathbb{R}^{2N}$. Physical properties of the system are represented by subsets of $\mathcal{S}$. Logical operations over physical properties – negation, disjunction and conjunction – are represented by the corresponding set-theoretic operations – complement, union and intersection – over subsets of $\mathcal{S}$. For instance, if the property 'The energy is between 10J and 12J' is represented by a subset X of $\mathcal{S}$ and the property 'The energy is between 12J and 15J' is represented by a subset Y of $\mathcal{S}$, then the property 'The energy is between 10J and 15J' is represented by the union $X \cup Y$.

In addition to logical operations, two relations among properties are defined. Two properties x and y are called *mutually exclusive* if they cannot both be true at the same instant. We say that x *implies* y if y is true whenever x is true. According to classical physics, two properties are mutually exclusive if the corresponding subsets are disjoint and x implies y if the subset of $\mathcal{S}$ that corresponds to y contains the subset of $\mathcal{S}$ that corresponds to x.

Consider a classical physical system with state space $\mathcal{S}$. In the following I will denote by $\mathcal{L}_c(\mathcal{S})$ the *lattice of physical properties* of this system, i.e. the set of subsets of $\mathcal{S}$ equipped with the logical operations, which turns $\mathcal{L}_c(\mathcal{S})$ into a Boolean lattice.

The mathematical framework for representing physical properties of quantum system was formulated by von-Neumann. A quantum physical system there corresponds to a separable Hilbert space $\mathcal{H}$. Properties are represented by closed subspaces of $\mathcal{H}$. The *lattice of physical properties* of the system that corresponds to the Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{L}_q(\mathcal{H})$, That is, $\mathcal{L}_q(\mathcal{H})$ is the set of closed subspaces of $\mathcal{H}$ with logical operations defined as follows: The negation of a property is represented by its orthogonal complement, and orthogonal subspaces represent mutually exclusive properties. For a pair of properties $X, Y \in \mathcal{L}_q(\mathcal{H})$ we say that X implies Y if $X \subseteq Y$.

Before defining binary operations between properties, we need to define the notion of *compatible properties*, which has no counterpart in classical physics[1]. Two properties $X, Y \in \mathcal{L}_q(\mathcal{H})$ are called compatible iff $\Pi_X \Pi_Y = \Pi_X \Pi_Y$ where Π_X and Π_Y are the orthogonal projection on X and Y, respectively. We define the disjunction of compatible properties as their algebraic span and the conjunction as their intersection. Note that mutually exclusive properties are always compatible. The operations defined above turn $\mathcal{L}_q(\mathcal{H})$ into an *orthomodular lattice*. The following table summarizes the similarities and difference between classical and quantum representations of properties.

Logic	Classical Rep.	Quantum Rep.
Propositions	Subsets of $\mathcal{S}$	Subspaces of $\mathcal{H}$
FALSE	ϕ	$\{0\}$
TRUE	Ω	$\mathcal{H}$
negation	X^c	$X^\perp$
logical implication	$X \subseteq Y$	$X \subseteq Y$
mutual exclusiveness	$X \cap Y = \phi$	$X \perp Y$
disjunction	$X \cup Y$	$X + Y$

Table 1: Physical properties

Note that we only define logical operations between subspaces with commuting projections. Adherents of the so called realistic quantum logic school, already explicit in von-Neumann and Birkhoff‘s paper [1], extend the scope of these operations, to every pairs of operators. However, as we shall see, restricting logical operations to compatible properties is sufficient for the derivation of subjective probability.

[1]More accurately, in classical physics all properties are compatible

Modeling Information: Probabilities and Beliefs

In classical physics, information about the state of a physical system with a phase-space $\mathcal{S}$ is represented by a probability distribution over $\mathcal{S}$, i.e. a real-valued function P over $\mathcal{L}_q(\mathcal{S})$ such that[2]

1. $\mathbb{P}(\mathcal{S}) = 1$
2. $\mathbb{P}(X) \geq 0$ for every $X \subseteq \mathcal{S}$
3. $\mathbb{P}(X \cup Y) = \mathbb{P}(X) + \mathbb{P}(Y)$ if $X \cap Y = \varnothing$.

In the abstract language of physical properties, the function $\mathbb{P}$ attaches a real-valued number – called probability – to every property, such that the probability of the disjunction of two mutually exclusive properties is the sum of their probabilities.

Quantum probabilities are defined similarly, replacing the classical representation of property with the quantum one (Recall table). Let $\mathcal{H}$ be a separable Hilbert space, corresponding to some quantum physical system. A *quantum probability over* $\mathcal{H}$ is a real valued function P over subspaces of $\mathcal{H}$ such that

1. $\mathbb{P}(\mathcal{H}) = 1$
2. $\mathbb{P}(X) \geq 0$ for every subspace X of $\mathcal{H}$
3. $\mathbb{P}(X \cup Y) = \mathbb{P}(X) + \mathbb{P}(Y)$ for every pair X, Y of orthogonal subspaces.

Gleason's Theorem characterizes all the quantum probabilities over a separable Hilbert space.

Gleason's Theorem. *Let $\mathcal{H}$ be a separable Hilbert space such that $\dim(\mathcal{H}) \geq 3$ and $\mathbb{P}$ a quantum probability over $\mathcal{H}$. Then there exists a trace-1 positive semi-definite operator ρ over $\mathcal{H}$ such that $p(X) = tr(\rho\Pi_X)$ for every subspace X where Π_X is the orthogonal projection over X.*

One important consequence of Gleason's Theorem is that there exists no quantum probability $\mathbb{P}$ over $\mathcal{H}$ such that $\mathbb{P}(X) = 0$ or $\mathbb{P}(X) = 1$ for every subspace X of $\mathcal{H}$: The observer can never know (with probability 1) the validity of every property of the system. This is in sharp contrast with the classical case.

Savage's Theory of Subjective Probability

From subjective probability perspective, the probability of a physical property is the amount of belief that the observer attaches to the validity of that property. This can be interpreted operationally as the amount of money that the observer is willing to invest in a gamble that offers a fixed prize should it turn out (after performing a suitable measurement) that the

[2]Omitting details about measurability and countable-additivity

property is true. Savage's approach to subjective probability [6] derives probabilities from a weak order over properties which represents the observer preference relation over gambles. This weak order satisfies several assumptions, which essentially says that the agent is, in some sense, rational.

We can apply Savage's Theorem to the lattice of properties of some classical physical system with state space $\mathcal{S}$. Let $\preceq$ be some weak order over $\mathcal{L}_c(\mathcal{S})$. The theorem provides sufficient conditions for the existence of a classical probability distribution $\mathbb{P}$ over $\mathcal{S}$ that *represents* $\preceq$, i.e. such that $X \preceq Y$ iff $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for every $X, Y \in \mathcal{L}_c(\mathcal{S})$. Consider, for example *de-Finetti's Axiom*. It says that if $X, Y, Z \in \mathcal{L}_c(\mathcal{S})$ such that $X \cap Z = Y \cap Z = \phi$ then $X \preceq Y \leftrightarrow X \cup Z \preceq Y \cup Z$. In terms of logical operations between physical properties, de-Finetti's axiom states that if x, y, z are properties such that the pairs x, z and y, z are mutually exclusive then $x \preceq y \leftrightarrow x \vee z \preceq y \vee z$. In [3] we use this same axiom to derive a representation of a weak order over $\mathcal{L}_q(\mathcal{H})$ by a quantum probability distribution over $\mathcal{H}$. Note that in de-Finetti axiom the disjunction of properties is only used for mutually exclusive properties, which are always compatible.

Bi-partite Systems

By a *bipartite physical system* is meant a physical system with two parts. For my purposes, it is best to think about the two parts of the system as located in two separate spots in space. In this section I describe the mathematical framework for handling bi-partite systems, first in classical physics and then in quantum physics. Especially I describe how a property of one part of the system is represented as a property of the bi-partite system, and how the state of the bi-partite system determines the state of each part.

Consider for example a physical system that is composed of two particles, x and y. A statement about one of the particles, such as 'x has energy between 10J and 15J', can be viewed as a property of the system that is composed of particle x alone or as a property of the bi-partite system that is composed of both particles. A physical theory must provide the observer with a procedure of translating every property of the x system to a property of the bi-partite xy-system

Bi-partite systems in classical physics

Consider a pair of classical physical systems with phase spaces A and B. If we view the bi-partite system as a single physical system then according to classical physics the phase space is given by $A \times B$. Let $\mathcal{L}_c(A)$ be the lattice of properties of the A-system. Consider a property $X \in \mathcal{L}_c(A)$ of the first system, i.e. $X \subset A$. Then, according to classical physics, the corresponding property of the bi-partite system is the element of $\mathcal{L}_c(A \times B)$ given by $X \times B$.

Let $\mathbb{P}$ be the state of the bipartite system. Recall that $\mathbb{P}$ is a probability distribution over $A \times B$ and that $\mathbb{P}$ is subjective: It summarizes the information of some observer about the bi-partite system. Denote by $\mathbb{P}_A$ the state of the A-system according to the same observer.

Then $\mathbb{P}$ dictates $\mathbb{P}_A$: For every $X \subset A$ one must have

$$\mathbb{P}_A(X) = \mathbb{P}(X \times B).$$

This follows from the identification of the property X of the A-system with the property $X \times B$ of the bi-partite system. Thus $\mathbb{P}_A$ is the marginal distribution of $\mathbb{P}$ over A.

Bi-partite systems in quantum physics

Consider a pair of classical physical systems with corresponding Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$. If we view the bi-partite system as a single physical system then according to quantum physics this bi-partite system lives in the Hilbert space $\mathcal{H} \otimes \mathcal{H}_B$ (the tensor product). Let $\mathcal{L}_q(X)$ be the lattice of properties of the A-system. Consider a property $X \in \mathcal{L}_c(A)$ of the first system, i.e. a subspace X of $\mathcal{H}$. Then, according to classical physics, the corresponding property of the bi-partite system is the element of $\mathcal{L}_c(A \times B)$ given by $X \otimes \mathcal{H}_B$.

The transformation from state of the bi-partite system to state of the subsystem is completely analogous to that of the classical setup. Let $\mathbb{P}$ be the state of the bipartite system, i.e. a quantum distribution over $\mathcal{H}_A \times \mathcal{H}_B$. Then the state $\mathbb{P}_A$ of the A-system is the quantum probability over $\mathcal{H}_A$ that is given by

$$\mathbb{P}_A(X) = \mathbb{P}(X \otimes \mathcal{H}_B).$$

$\mathbb{P}_A$ is called the marginal distribution over $\mathcal{H}_A$.

Signaling and Blackwell's Theorem

Bipartite physical systems serves as a natural framework for *signaling*: An agent gains information about one part of a bi-partite physical systems by performing measurements over the other part. In this section I formulate Blackwell's theory of comparison of statistical experiments [2] in terms of classical physical systems. I prove an analogue theorem in the quantum setup in [7].

A *classical information structure* is given by a triple (N, S, p) where N and S are finite sets and p is a (classical) distribution over $N \times S$, i.e. $p = (p_{\{n,s\}})_{n \in N, s \in S}$ such that $p_{n,s} \geq 0$ and $\sum_{n,s} p_{n,s} = 1$. We can think of N and S as the phase space of two classical physical particles.

A *game* is given by a finite set A whose elements are called *actions*, each action $a \in A$ corresponding to a payoff function $M^a : N \to \mathbf{R}$. The game is played as follows: First a pair $(n, s) \in N \times S$ is randomly chosen according to p. The observer sees s (the state of particle S) and then chooses an action $a \in A$. The observer's payoff is given by $M^a(n)$. In terms of physics, every action correspond to some physical measurement that is carried on over the N-system and the numerical outcome of this measurement is the payoff of the observer. We assume that the observer is *rational*, i.e. that he chooses his action in according to a *strategy* that maximizes his expected payoff.

Let (N, S, p) and (N, T, q) be two information structures. (N, S, p) is said to be *better* than (N, T, q) if, for every game (that is, for every finite set A and every payoff functions

$\{M^a : N \to \mathbf{R}\}_{a\in A}$), the expected payoff to the rational observer if the game is played over (N, S, p) is at least as good as the expected payoff if the game is played over (N, T, q). Thus the partial order 'better' over information structures is defined in terms of games. Blackwell's Theorem characterizes the same order in purely probabilistic terms:

Theorem 0.0.1. *Let (N, S, p) and (N, T, q) be two classical information structures. Then (N, S, p) is better than (N, T, q) if and only if there exists a matrix $F = (f_{s,t})_{s\in S,t\in T}$ such that $f_{s,t} \geq 0$ and $\sum_t f_{s,t} = 1$ for every $s \in S$ (i.e. F is a stochastic matrix) and*

$$q_{n,t} = \sum_s p_{n,s} f_{s,t} \text{ for every } n, t.$$

Note that every stochastic matrix $F = (f_{s,t})_{s\in S,t\in T}$ corresponds to a linear transformation $\mathcal{F} : \mathbf{R}^S \to \mathbf{R}^T$ that transforms probability distributions over S to probability distributions over T. We call $\mathcal{F}$ a (classical) stochastic map. In statistical literature, the set S is viewed as a set of possible *signals* to the observer. The application of the stochastic map $\mathcal{F}$ in Blackwell's Theorem is interpreted as *simulation*: If the observer receives a signal s he creates, or simulates, a new signal t from the set T, distributed according to the s-th line of the matrix F. If the distribution of (n, s) was p, the simulation process results in a new signal t such that the joint distribution of (n, t) is q.

In terms of physics, stochastic maps correspond to all the *physical manipulations* that can be performed over a classical particles. The physical meaning of the above equation is that during the simulation process the observer performs manipulations only upon his part of the bipartite system.

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