Sample Report on

Off-Centers: A new Type of Steiner Points for Computing Size-Optimal Quality-Guaranteed Delaunay Triangulations

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Problem statement, solution, and discussion Delaunay refinement is a technique to produce a *graded* triangular mesh of a domain with piecewise-linear boundary. Starting with an initial coarse mesh, such as the Delaunay triangulation of the vertices of the input, we successively refine the triangulation by adding *Steiner points*. Typically, we insert the circumcenter of a triangle that is *skinny*, i.e., whose circumradius to shortest-edge-length ratio is greater than a fixed threshold parameter ρ. Inserting the circumcenter destroys the skinny triangle when the Delaunay triangulation is updated. There are two challenges with Delaunay refinement that impact its applicability:

1. Potentially a large number of Steiner points might need to be inserted before every triangle in the mesh has radius-edge ratio at most ρ, the quality parameter.
2. Each insertion might require a number of edge flips to restore the Delaunay property. Successive insertions might "collide" in the sense that they might trigger edge flips in overlapping parts of the domain. Therefore, successive insertions might have to be performed sequentially. This sequential dependence slows down the meshing algorithm in applications where parallel algorithms are often necessary.

Off-center insertion is proposed as an alternative to Delaunay refinement to reduce the final mesh size and to limit the overlap between different insertions in order to develop a parallel Delaunay refinement algorithm (which the author does in a separate paper: "Parallel Delaunay Refinement: Algorithms and Analyses" with Dan Spielman and Shang-Hua Teng).

Off-center insertion proposes to insert the so-called *off-center* of a skinny triangle $\triangle pqr$, defined as the third vertex c that, together with the shortest edge qr, defines a triangle with radius-edge ratio exactly equal to ρ. The off-center c lies on the segment joining the circumcenter of $\triangle pqr$ to the midpoint of qr.

Pros The main advantage of off-center insertion over circumcenter insertion is that it alleviates both problems listed above. The $\triangle cqr$ after off-center insertion is already of acceptable quality and does not trigger further refinement by itself. Further, the circumcircle of $\triangle cqr$ is mostly contained in the (empty) circumcircle of $\triangle pqr$ and therefore $\triangle cqr$ *may* already be Delaunay (though this is not guaranteed). Experiments by the author show that, as expected, the resulting triangulation has about 15% to 20%

fewer triangles than that by the classical circumcenter insertion algorithm. At the same time, all proofs of theoretical guarantees and finite termination carry over with trivial changes from those for circumcenter insertion.

Thus, the main impressive contribution of this paper is that it improves in practice dramatically over a popular classical algorithm in use for a long time while still retaining the theoretical guarantees of the classical algorithm.

Cons The chief concern I have about off-center insertion is that it might skew the distribution of quality of triangles towards the minimum acceptable threshold of ρ. Off-center insertion specifically creates triangles whose radius-edge ratio is exactly equal to ρ and these triangles might survive into the final mesh. On the other hand, circumcenter insertion might optimize the distribution better and avoid creating too many triangles whose quality is close to the lower bound of ρ. Thus, off-center insertion achieves a different tradeoff between mesh quality and mesh size. It is therefore important in off-center insertion to set the threshold ρ prudently, no smaller than necessary. It might be necessary to introduce an additional optimization step to try various values for ρ and choose the final value such that the mesh has acceptable minimum quality and does not have too many triangles.

Open problem I question why we choose to insert, instead of the off-center of $\triangle pqr$, the vertex e such that $\triangle eqr$ is *equilateral*. An equilateral triangle has the best possible ratio of circumradius to shortest edge length and it would not require further improvement. Further, its circumcircle is contained in the circumcircle of $\triangle cqr$ produced by off-center insertion and is therefore even more likely to be Delaunay (though again this is not guaranteed). I see only advantages to insertion of this so-called *equi-center* over that of the off-center. An experimental study, perhaps using the Triangle software by Shewchuk, is needed to evaluate the relative advantages of equi-center, off-center, and circumcenter insertion for Delaunay refinement.

Sample Homework Solution
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