

Fréchet Distance between Curves in the Plane with Obstacles

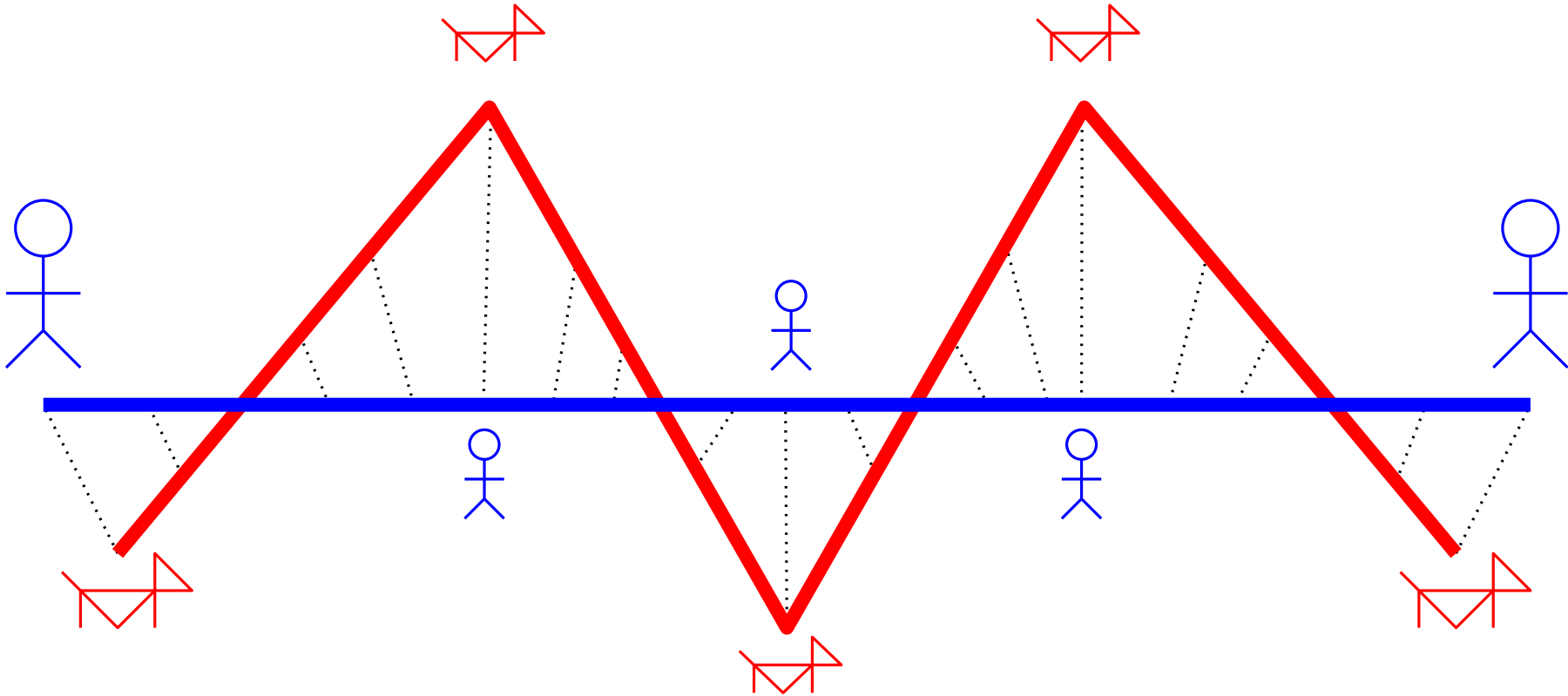
Shripad Thite

TU-Eindhoven

Joint work with

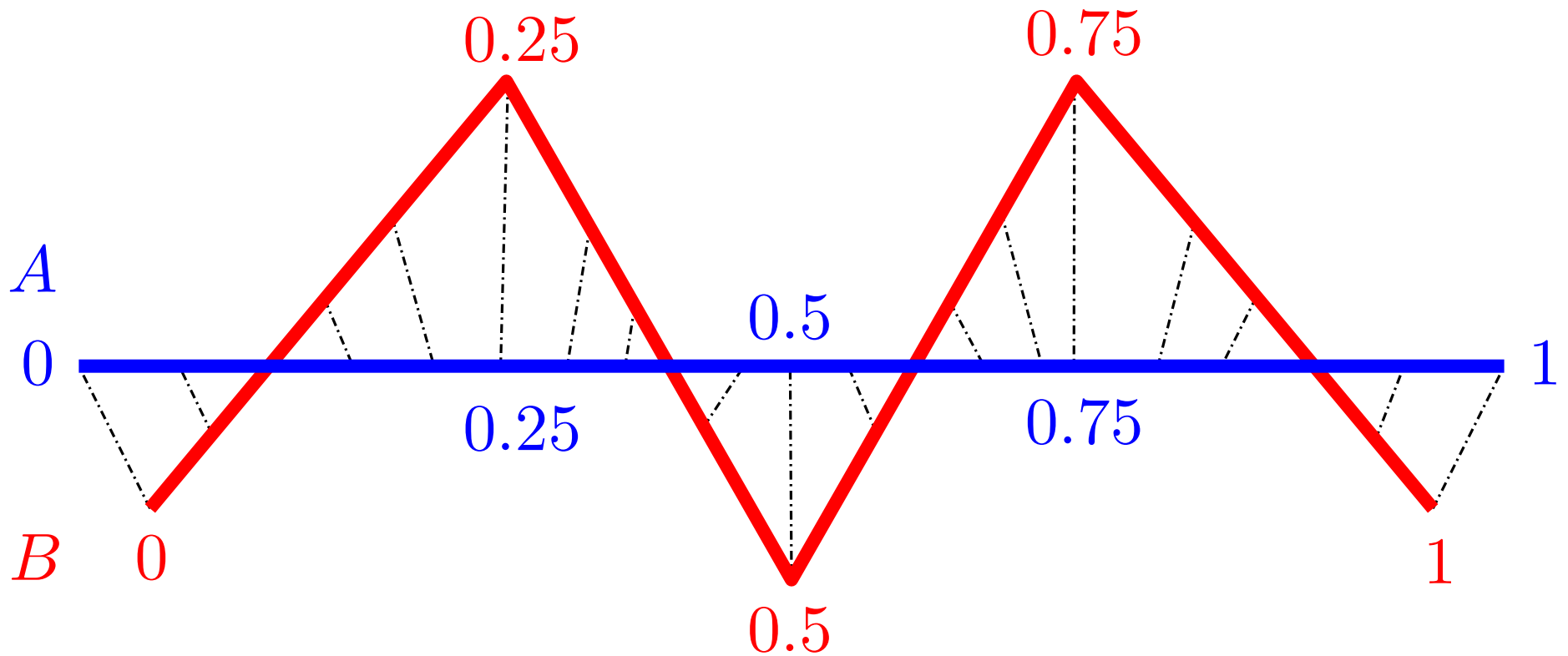
Erin Wolf Chambers, Éric Colin de Verdière, Jeff Erickson,
Sylvain Lazard, Francis Lazarus

Fréchet distance between curves



Dog-leash distance: Minimum length of a leash joining a **dog** and its **owner** as they walk along their respective curves, from one endpoint to the other, continuously without back-tracking

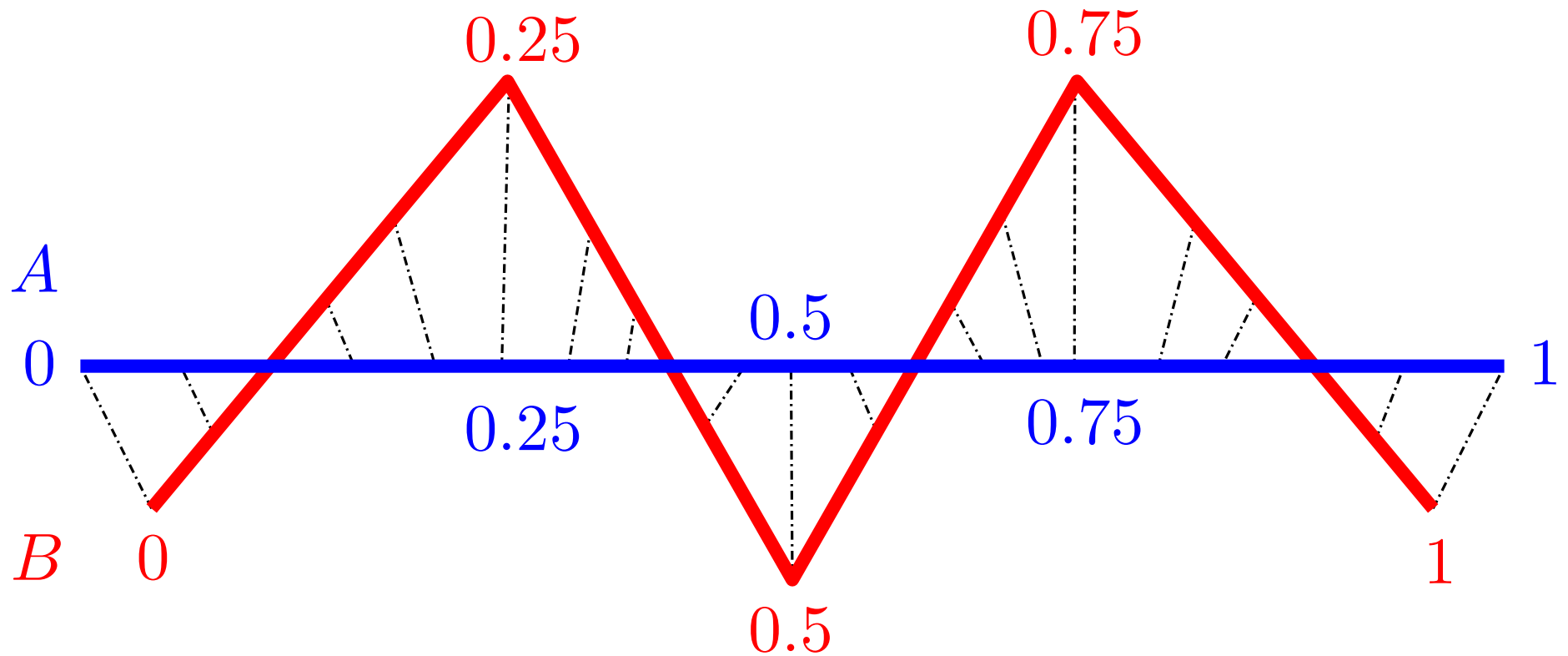
Fréchet distance between curves



Leash endpoints move continuously, monotonically
(better measure of similarity than Hausdorff distance)

Leash at time t is a straight line joining the point $A(t)$ and the point $B(t)$

Morphing between curves



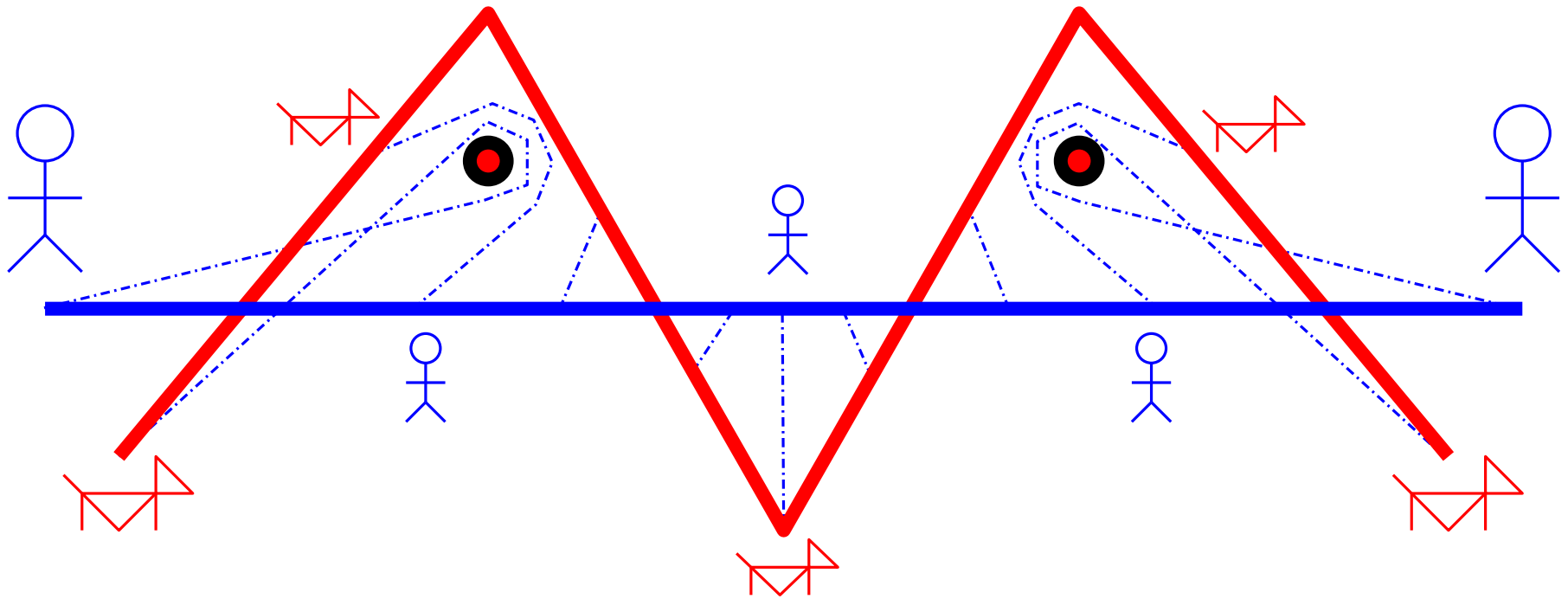
Leash motion encodes a continuous deformation between A and B

The “cost” of the deformation is the maximum distance any point has to travel, i.e., $\max_{t \in [0,1]} \text{dist}(A(t), B(t))$

Parks have trees

...and other obstacles

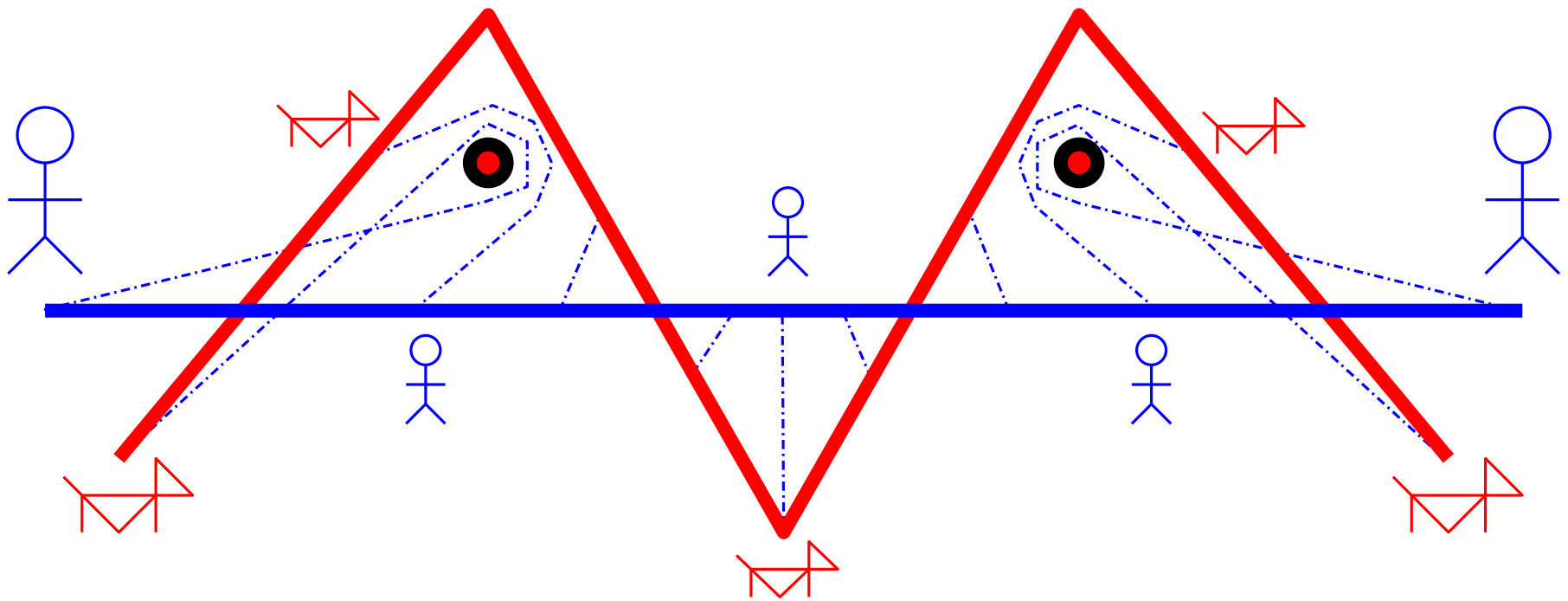
New condition: Leash cannot jump over obstacles



A longer leash may be required than without obstacles

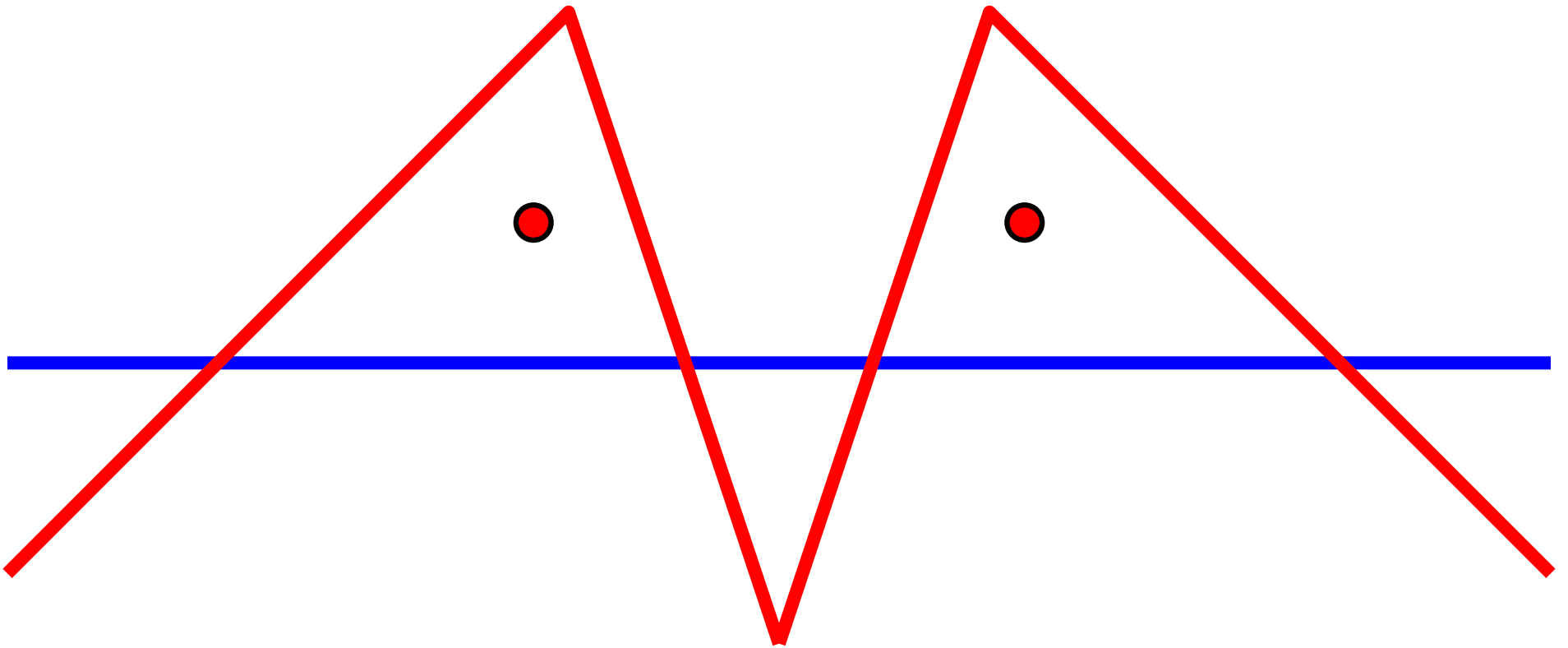
Homotopic Fréchet distance

Dog-leash distance where the leash cannot jump over obstacles; longer leash required to wind around obstacles

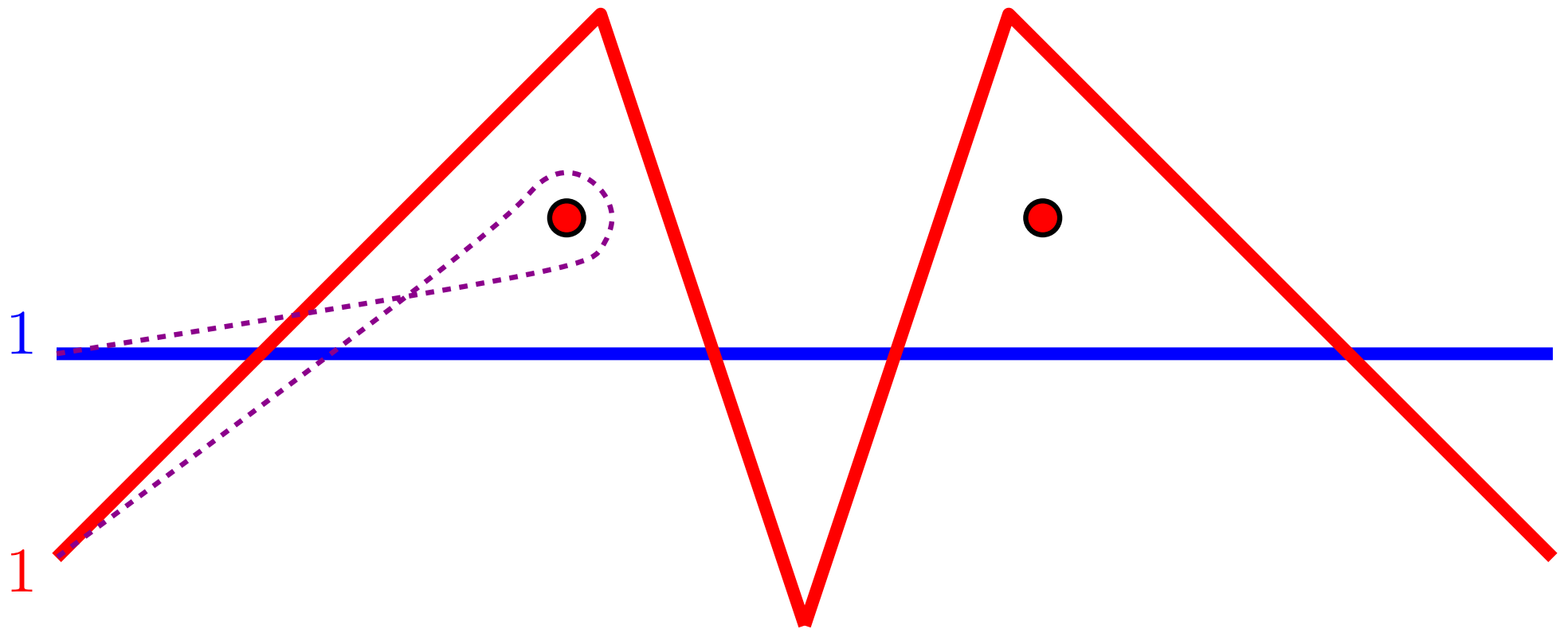


We give polynomial-time algorithms to compute the homotopic Fréchet distance between two polygonal curves in the plane with **point obstacles**

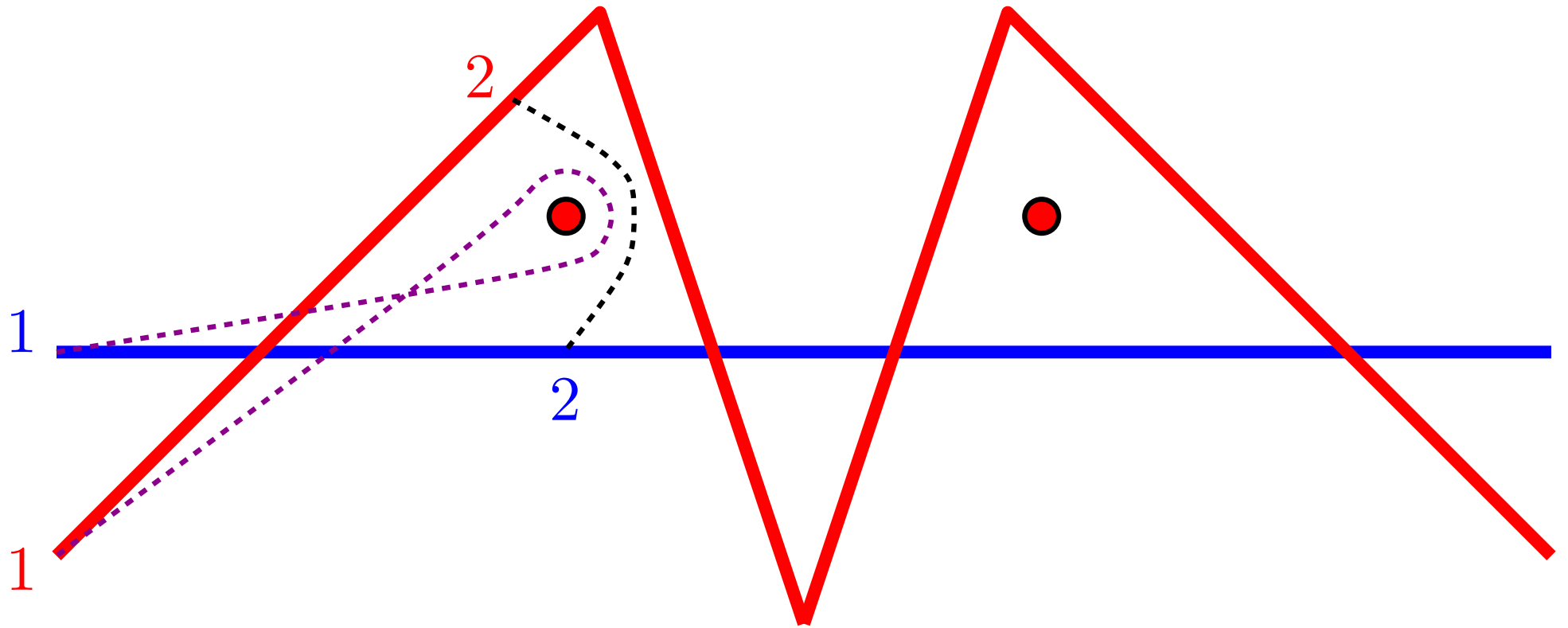
Example 1



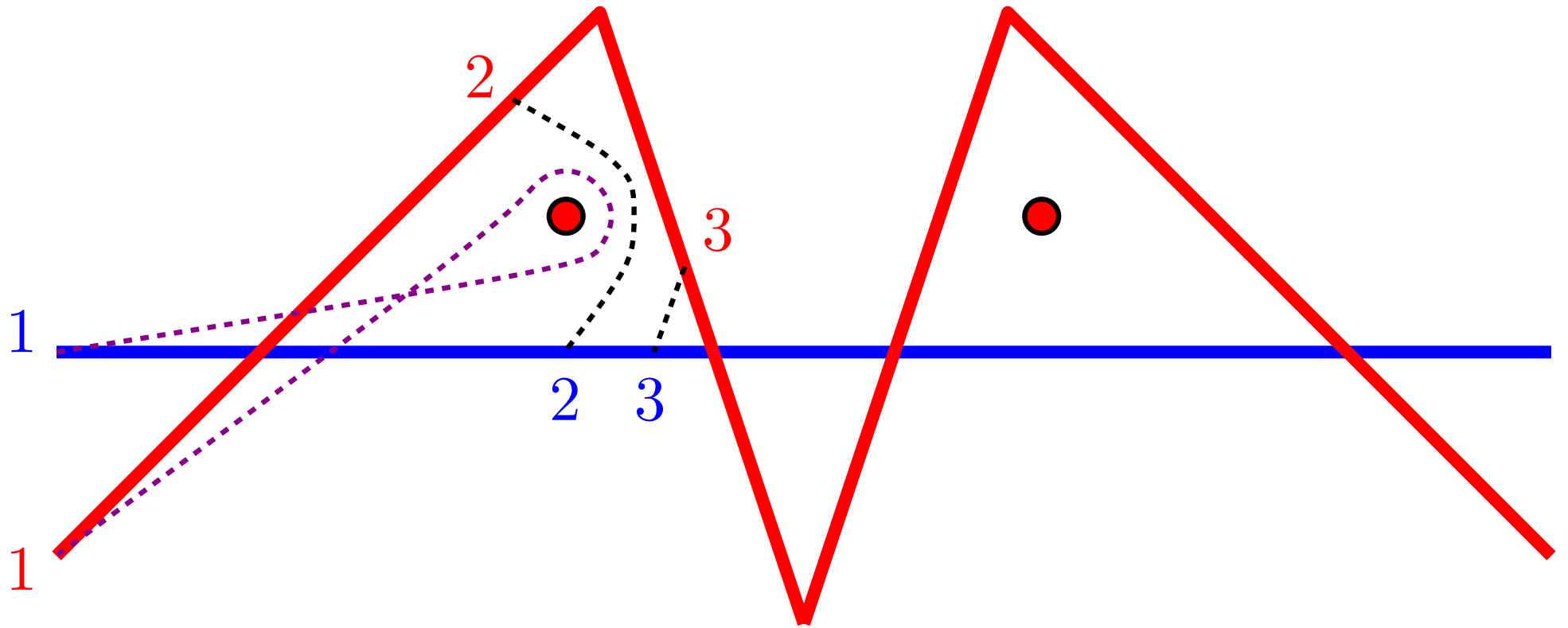
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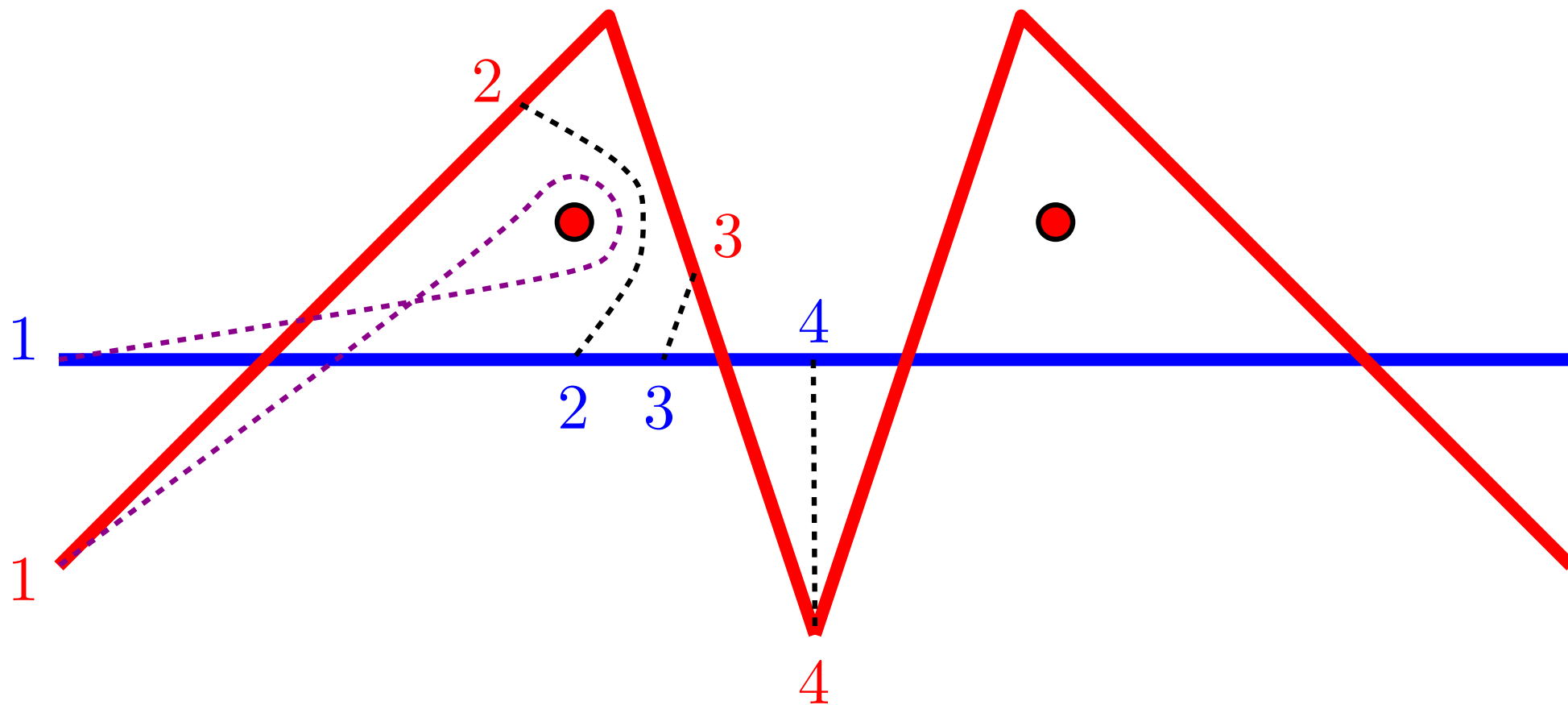
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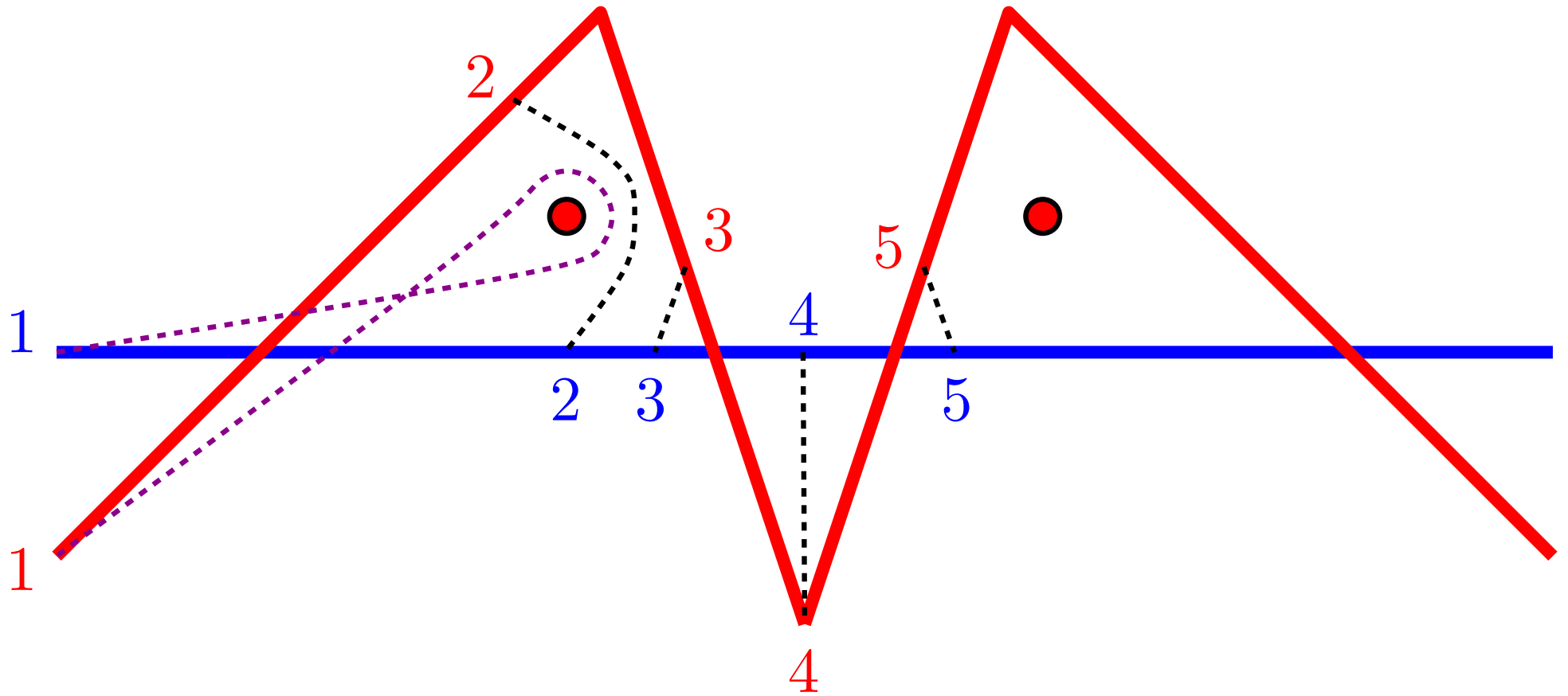
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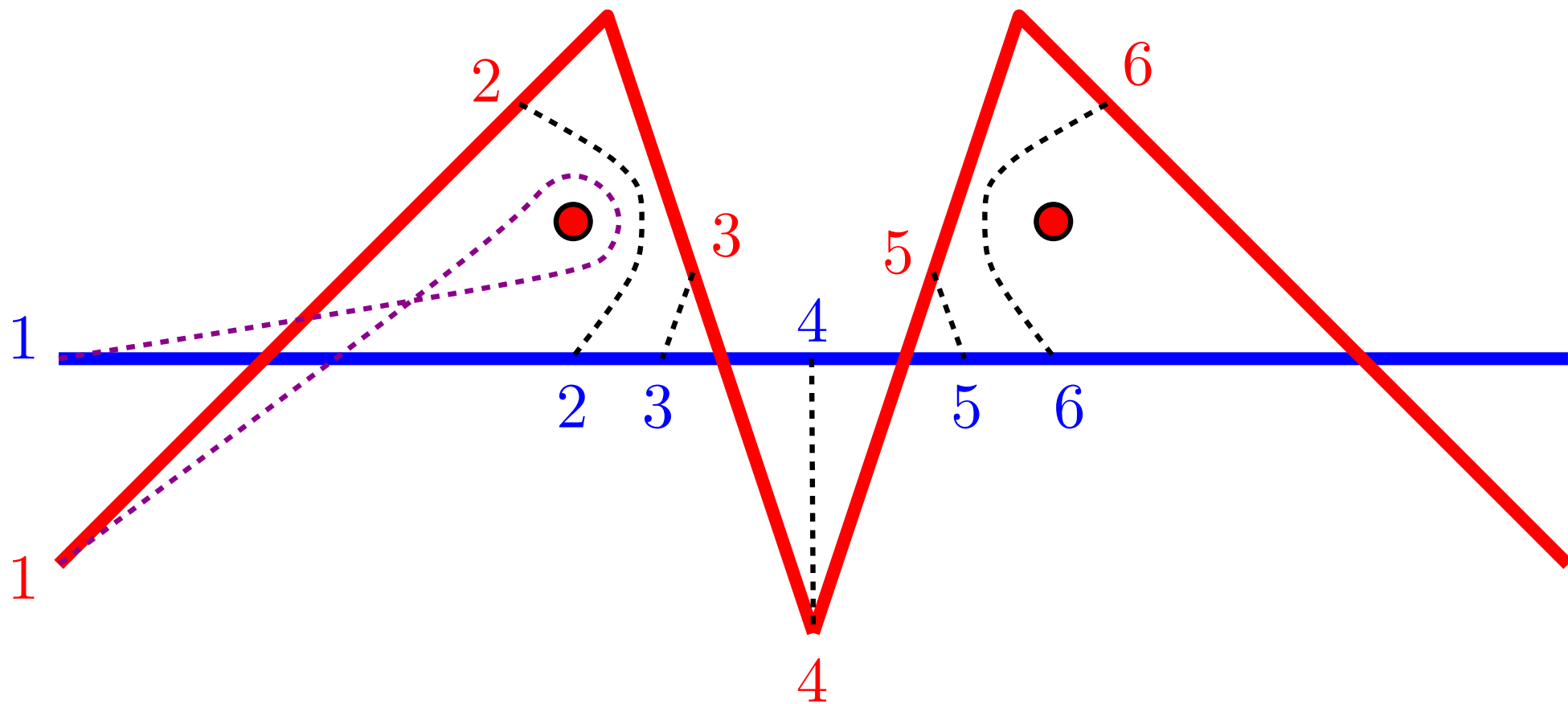
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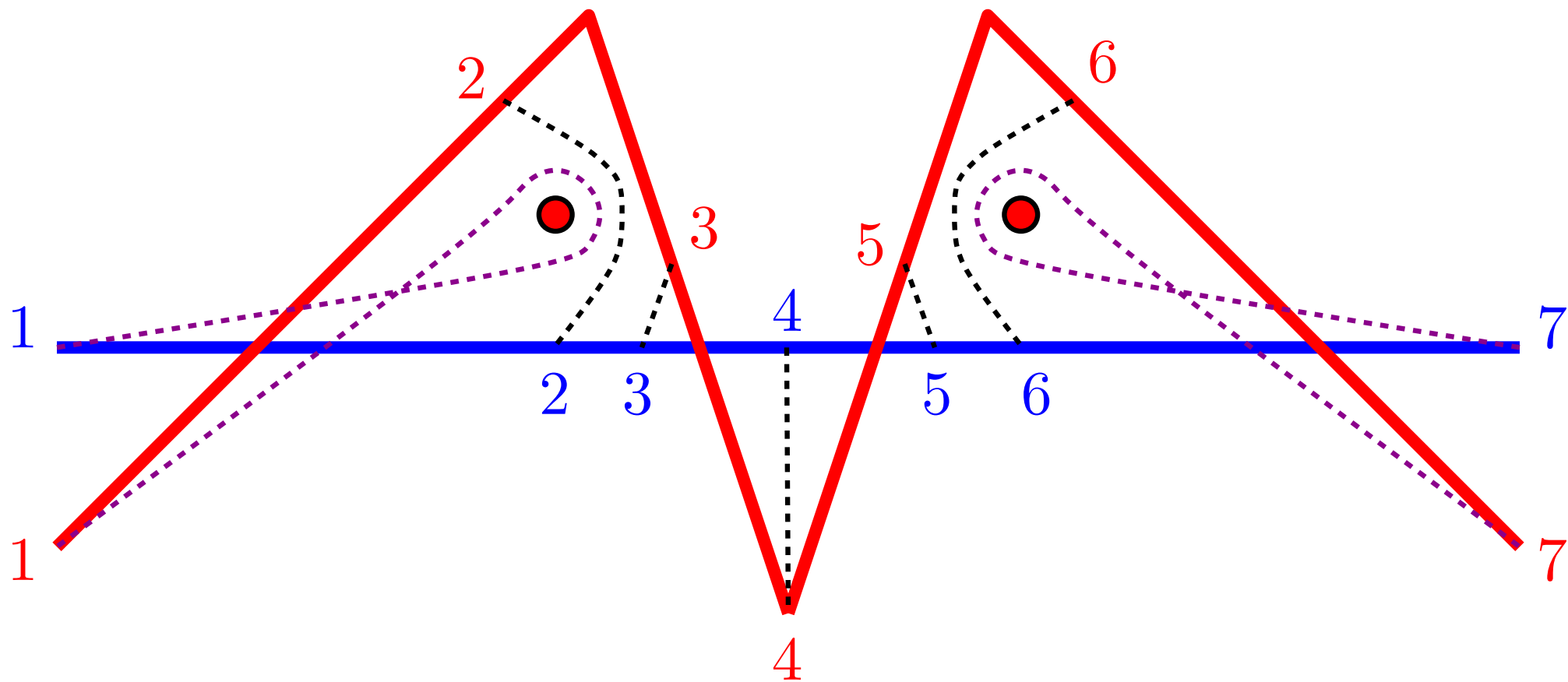
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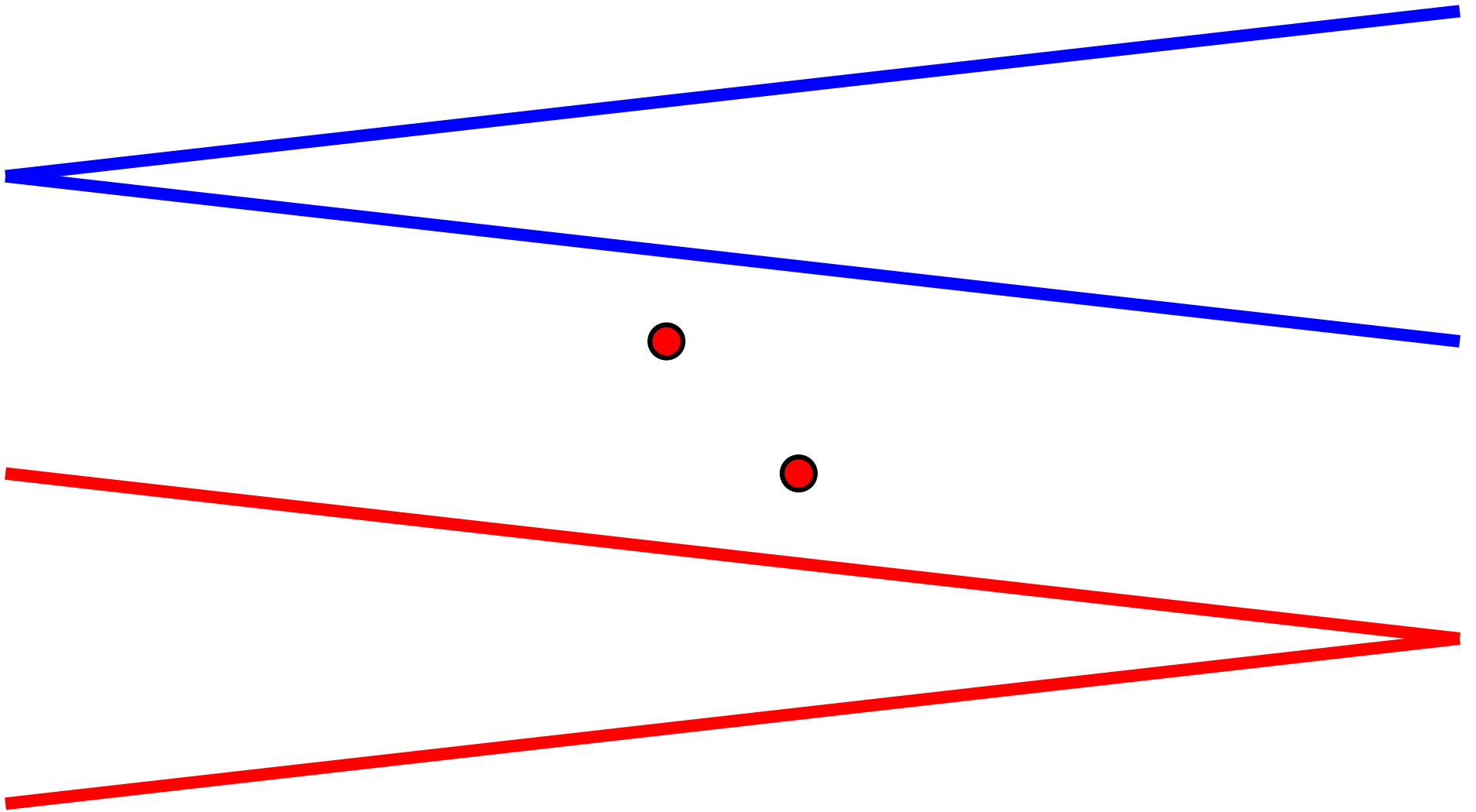
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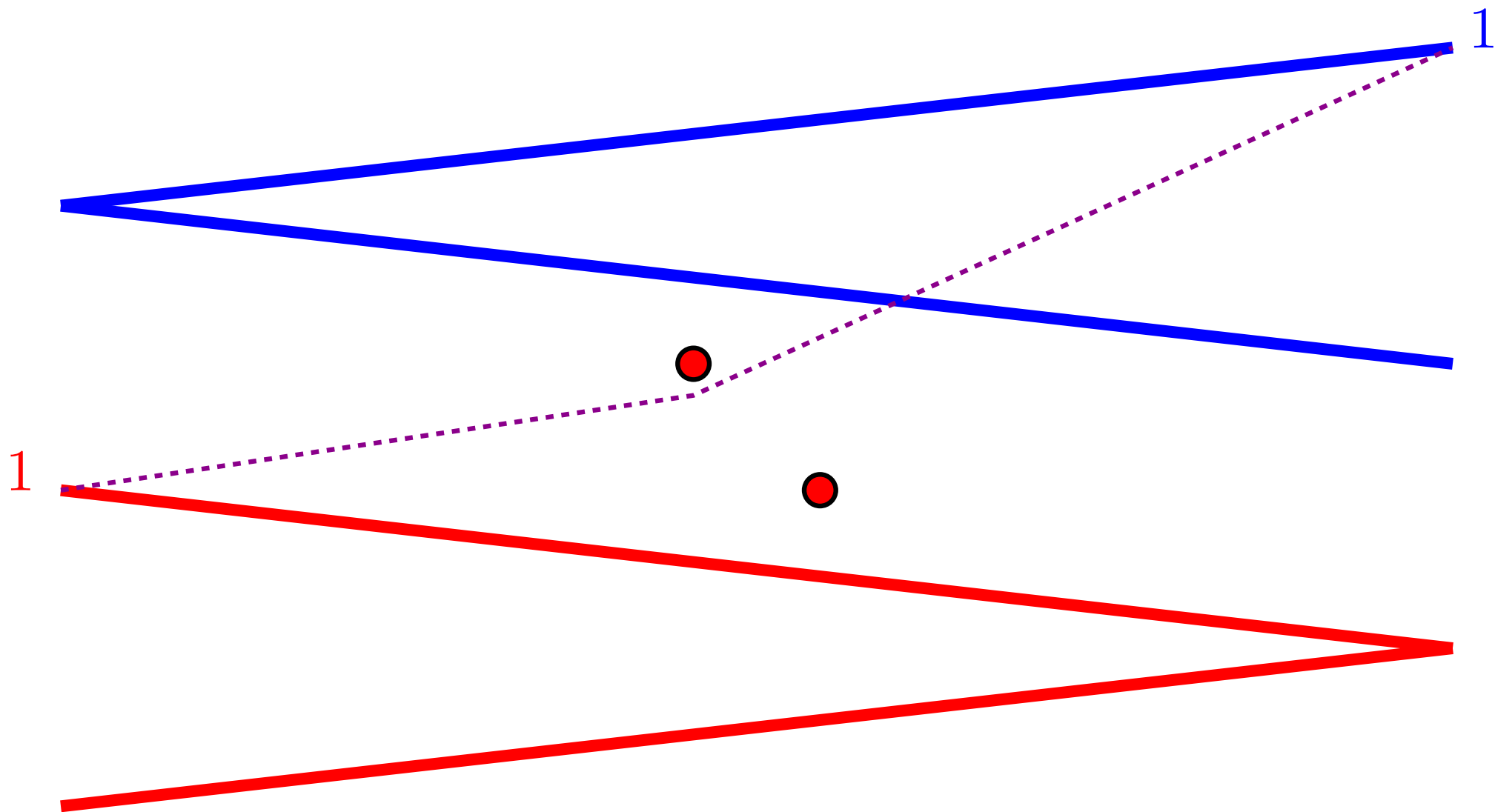
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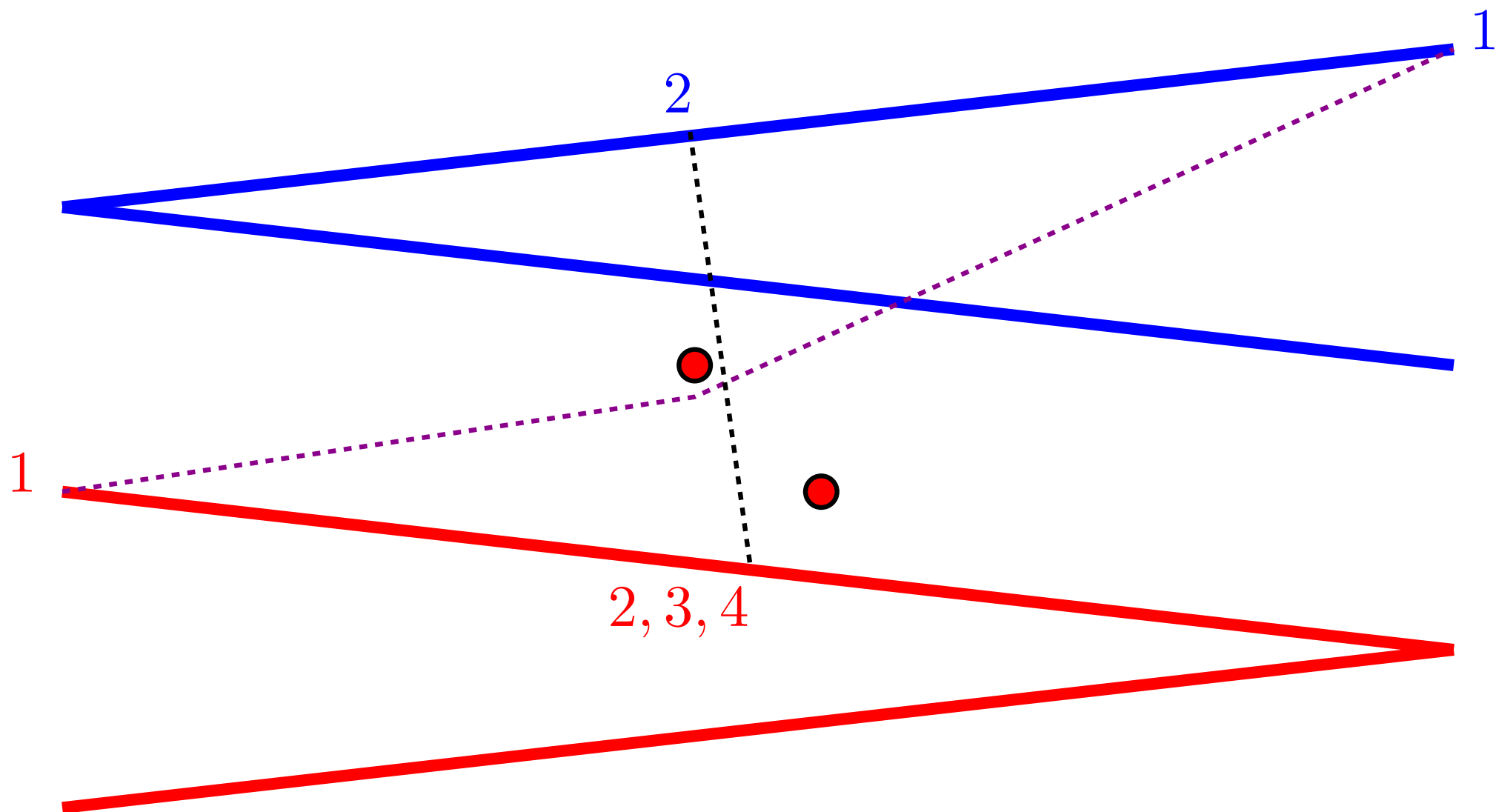
Example 2



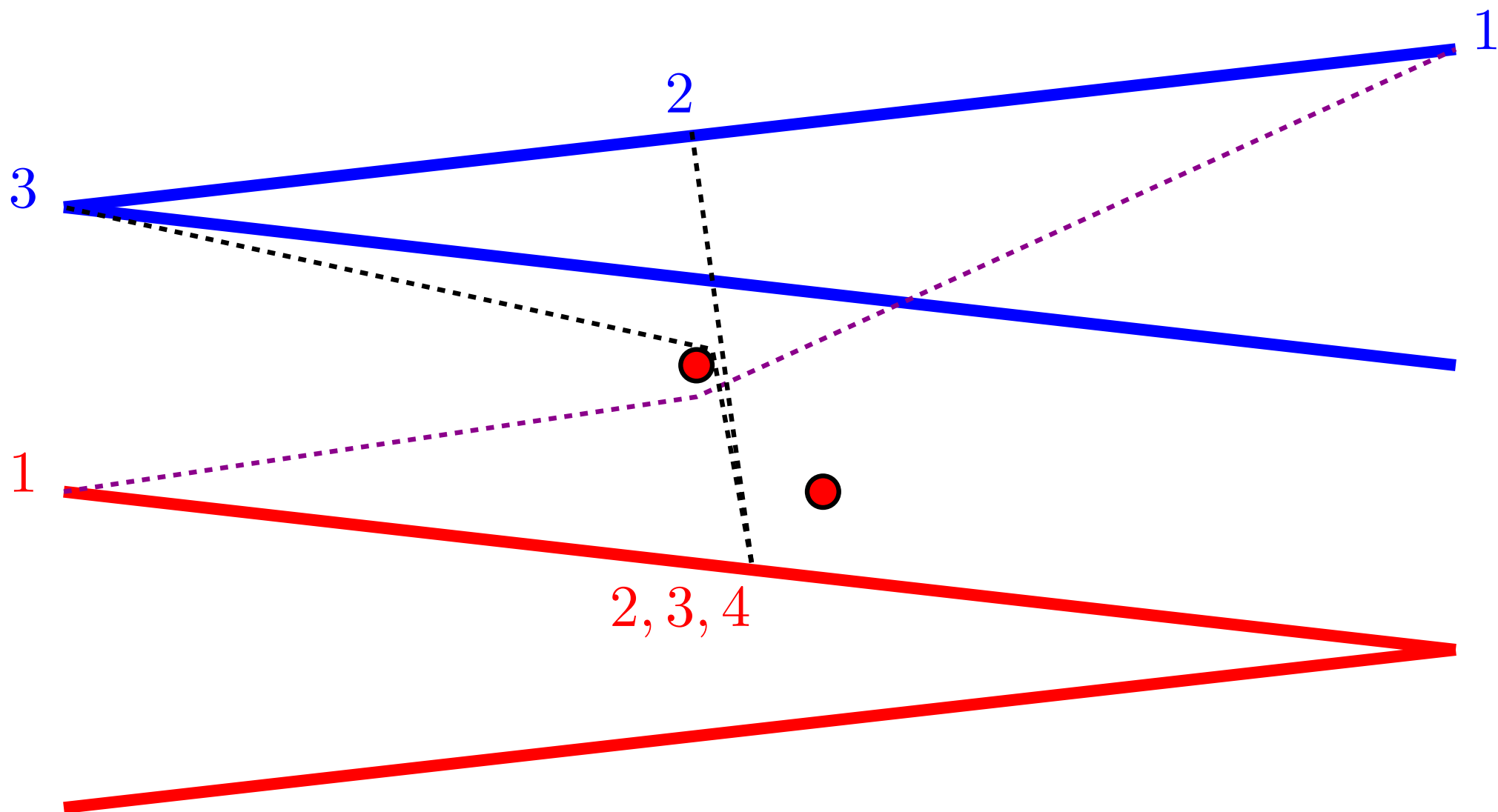
Example 2



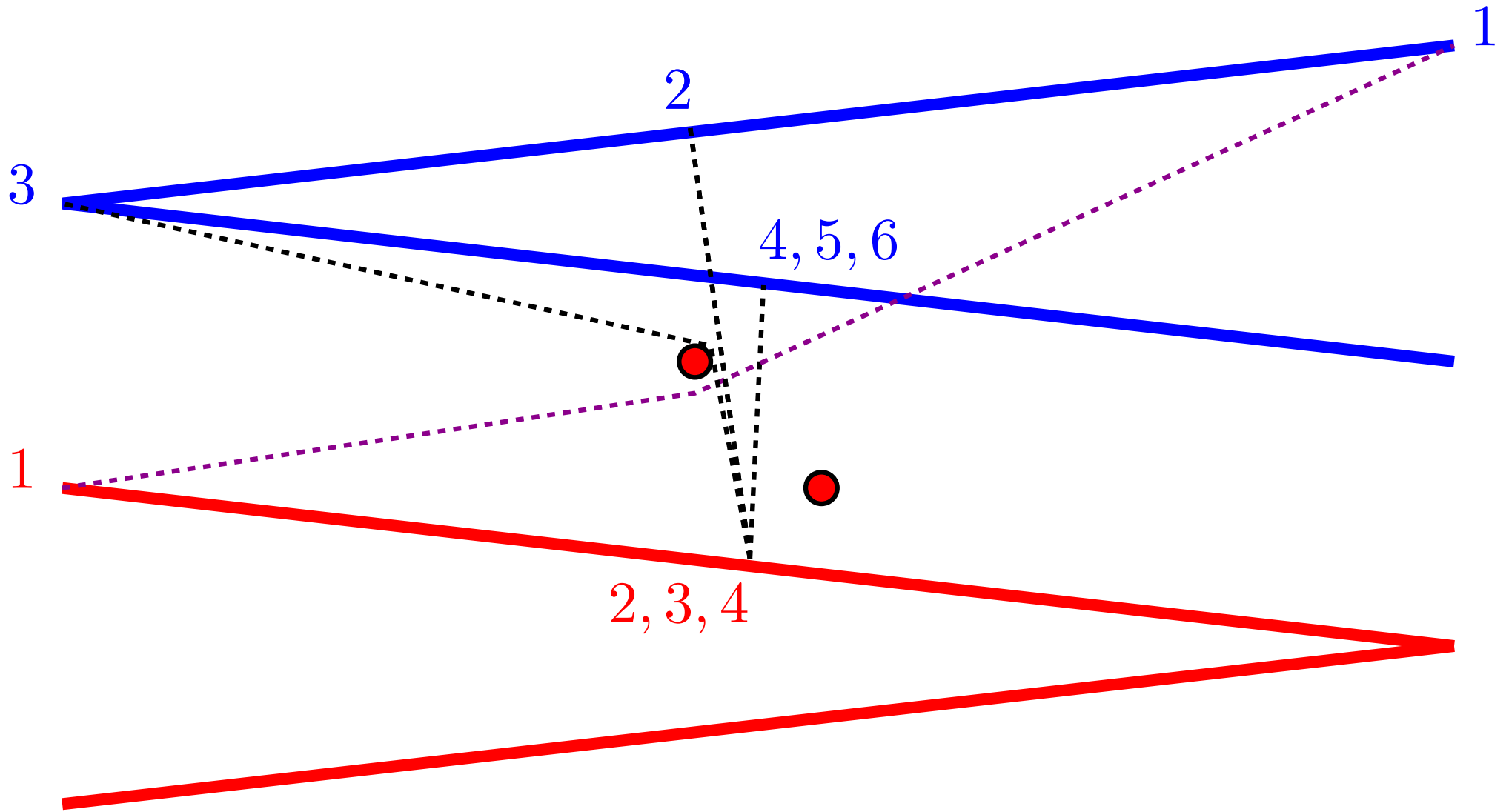
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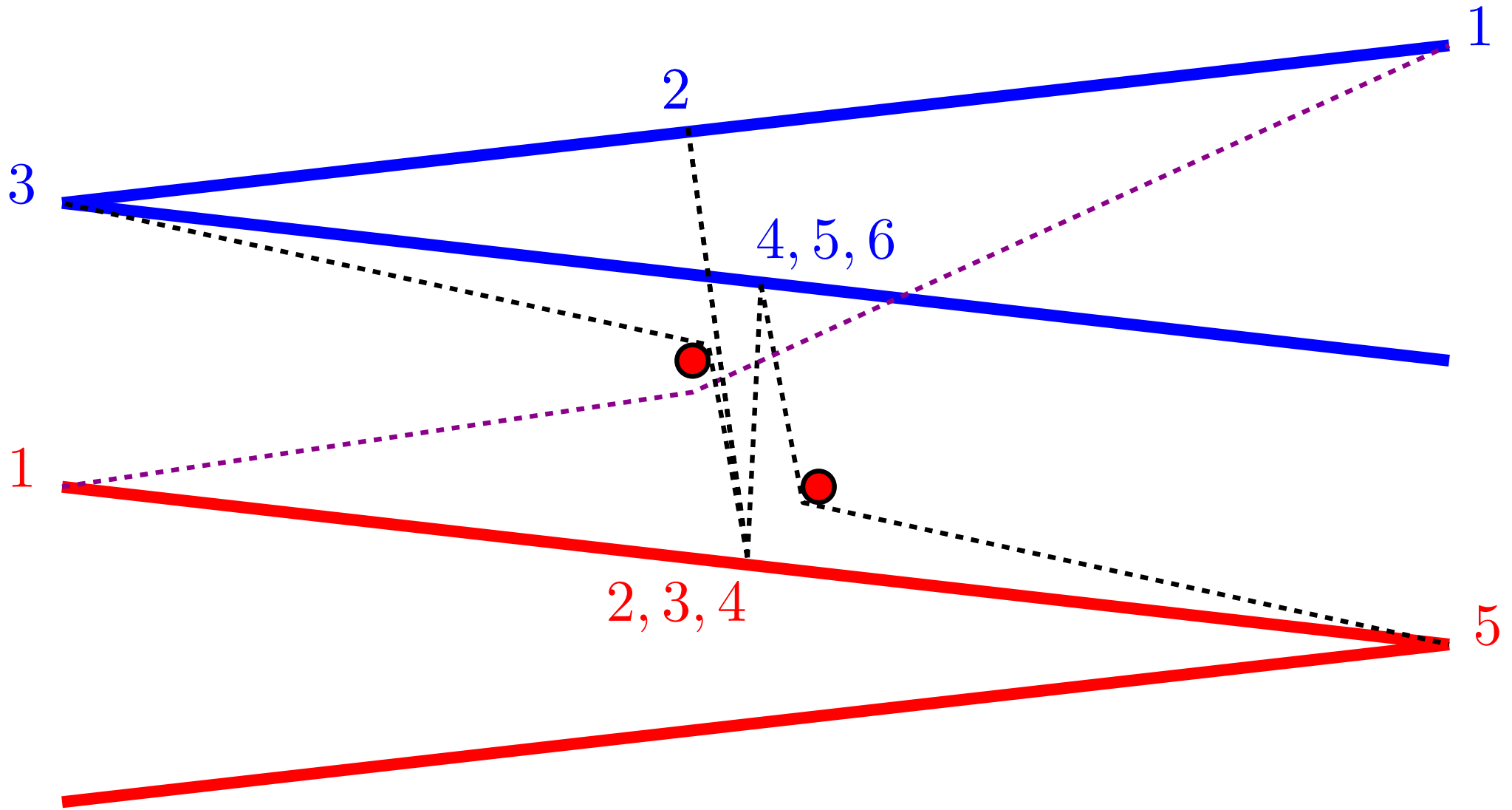
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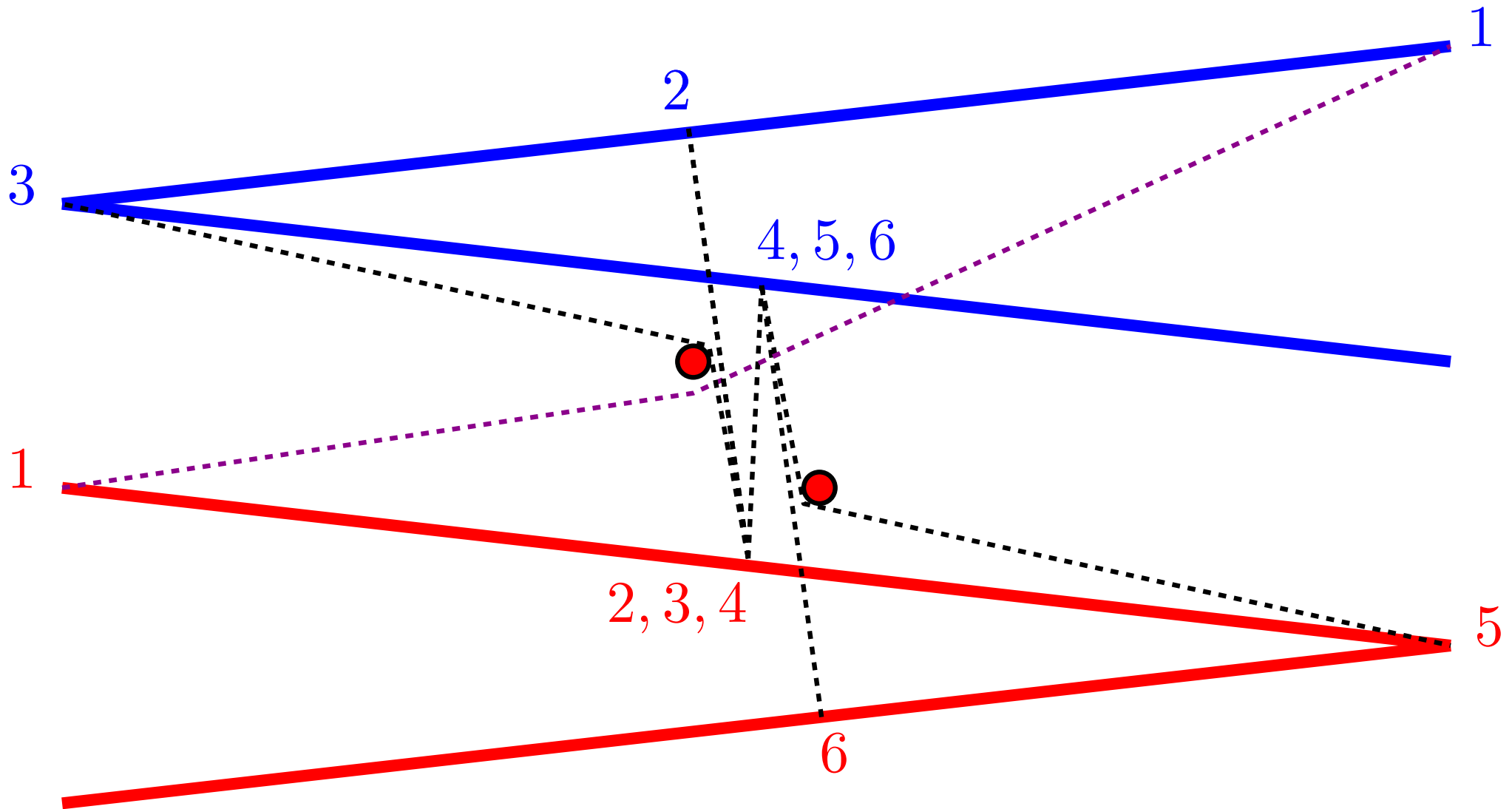
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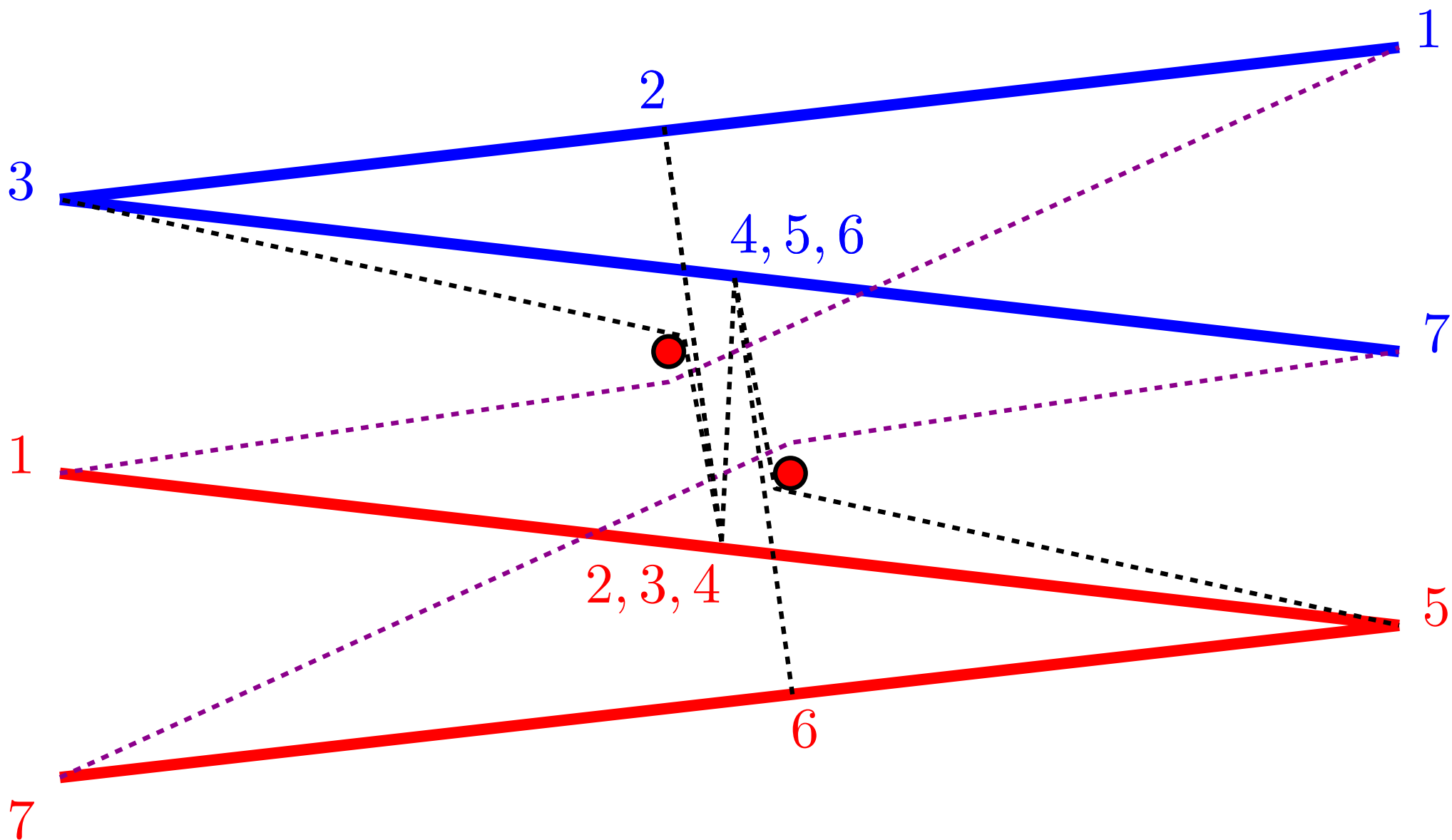
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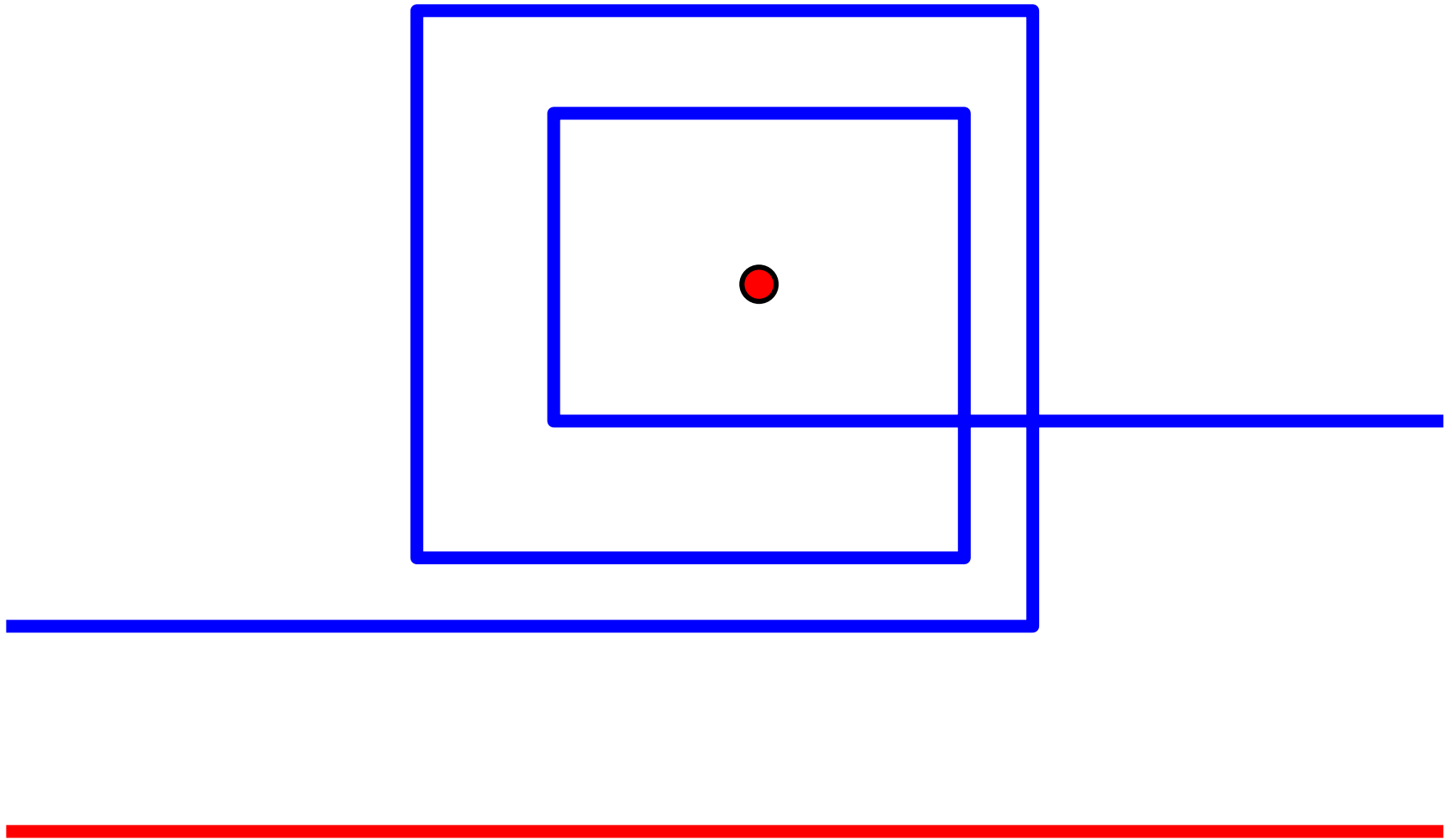
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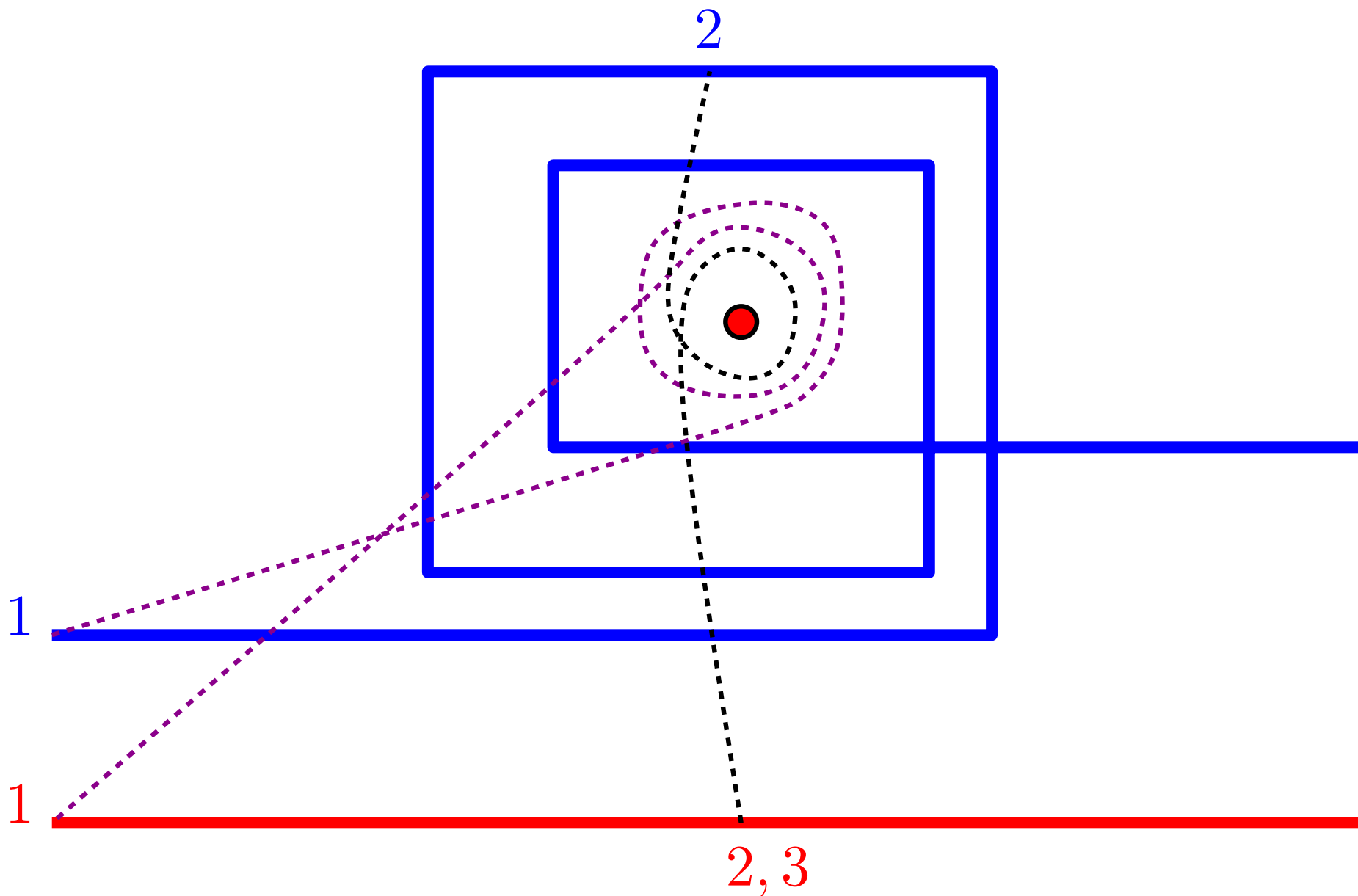
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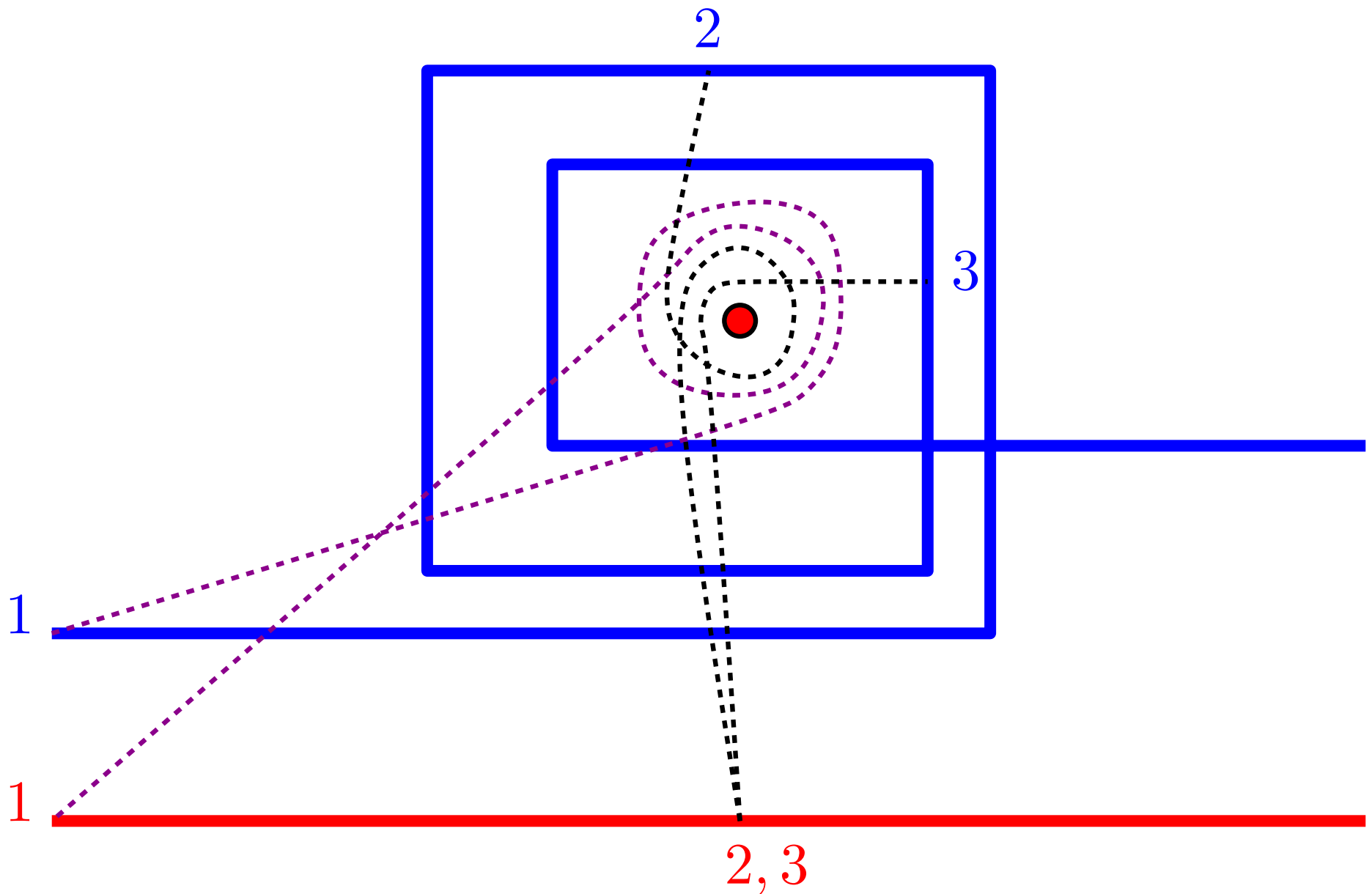
Example 3



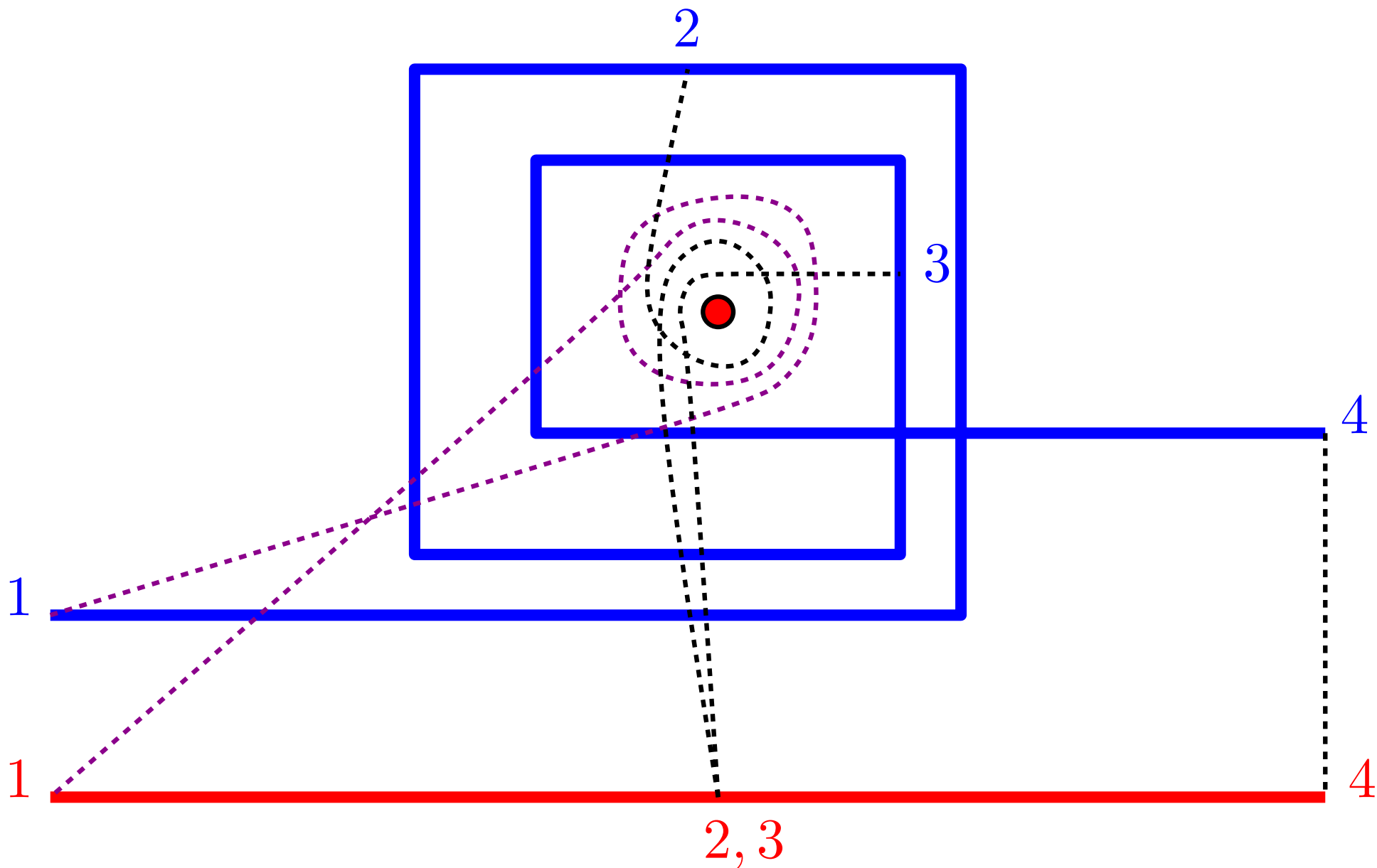
Example 3



Example 3



Example 3



Punctured plane

Let P be a set of points in the plane = obstacles

Punctured plane = $\mathbb{E}^2 \setminus P$

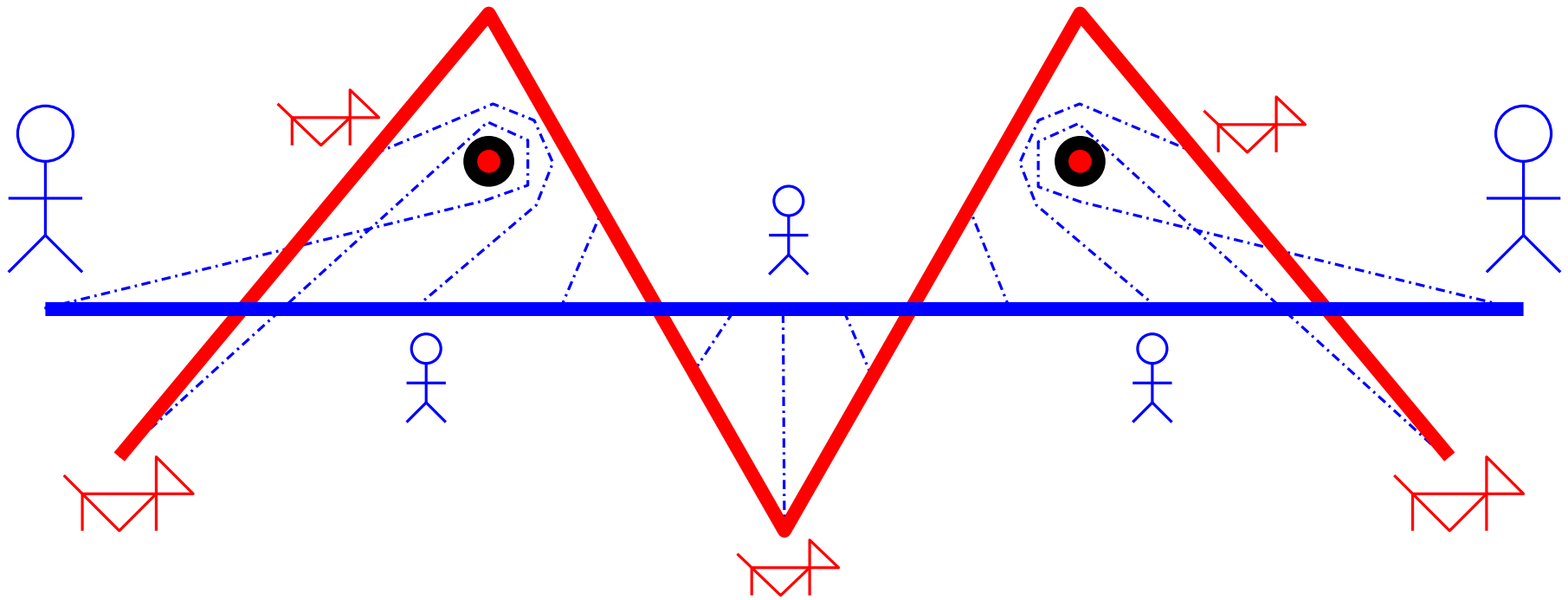
Leash must never intersect a point of P

Leash must never jump over an obstacle

Relative homotopy

A **leash** is a curve joining a point of A and a point of B

Two leashes are **relatively homotopic** if they are continuously deformable in the punctured plane *while keeping their endpoints on the respective curves*



Optimum leash motion = min-length relative homotopy

Geodesic leashes

Lemma: There exists an optimum leash motion such that the leash at every time is a shortest path in its homotopy class

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Let ℓ_h denote a leash motion in the homotopy class h

$$\text{cost}(\ell_h) = \sup_{t \in [0,1]} |\sigma_h(A(t), B(t))|$$

where $\sigma(a, b)$ is the shortest path between a and b in homotopy class h

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Optimum leash motion ℓ^* minimizes cost, over all homotopy classes

We allow the leash to pass through obstacle points. The definition of a homotopy class is suitably extended. Now, shortest paths exist in every homotopy class.

Key lemma

The optimum homotopy class h^* must contain a straight-line leash

i.e., there is no obstacle p such that the leash always bends around p

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Our strategy:

We enumerate all homotopy classes that contain a straight-line leash, i.e., a line segment joining some point of A and some point of B

There are a polynomial number of such minimal homotopy classes

Leash motion in homotopy class h

Decision problem: Given a real $\varepsilon \geq 0$, is there a leash map whose cost is at most ε ?

i.e., can the dog be walked with a leash of length ε ?

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The critical values of ε can be computed in polynomial time

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The critical values of ε can be computed in polynomial time

Optimization problem: Find the smallest ε for which the answer above is 'yes'

Parametric search

'nuff said!

Open problem

Generalize to polygonal obstacles

Thank you!

Extra slides

Leash map

Curves $A, B : [0, 1] \rightarrow \mathbb{E}^2$

A **leash map** is a *continuous* function

$$\ell : [0, 1] \times [0, 1] \rightarrow \mathbb{E}^2 \setminus P$$

arc-length parameterization of leash \uparrow time \uparrow

such that there exist re-parameterizations u and v of A and B where $\ell(0, \cdot) = A \circ u$ and $\ell(1, \cdot) = B \circ v$

$\ell(\cdot, t)$ is a curve in $\mathbb{E}^2 \setminus P$ joining the point $A(u(t))$ and the point $B(v(t))$

Optimum leash map

A leash map is a relative homotopy between leashes, so it belongs to a relative homotopy class

$$\ell_h : [0, 1] \times [0, 1] \rightarrow \mathbb{E}^2 \setminus P$$

such that $\ell_h(\cdot, t)$ is a leash in homotopy class h

$$\text{cost}(\ell_h) = \sup_{t \in [0, 1]} \{ \text{Euclidean length of } \ell_h(\cdot, t) \}$$

We want a minimum-cost leash map ℓ^*

$$\ell^* = \arg \min_h \{ \text{cost}(\ell_h) \}$$

Homotopic Fréchet distance = $\text{cost}(\ell^*)$

In case you were wondering . . .

$$O(n^8 \log n)$$