Walking Your Dog in the Woods in Polynomial Time

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Joint work with

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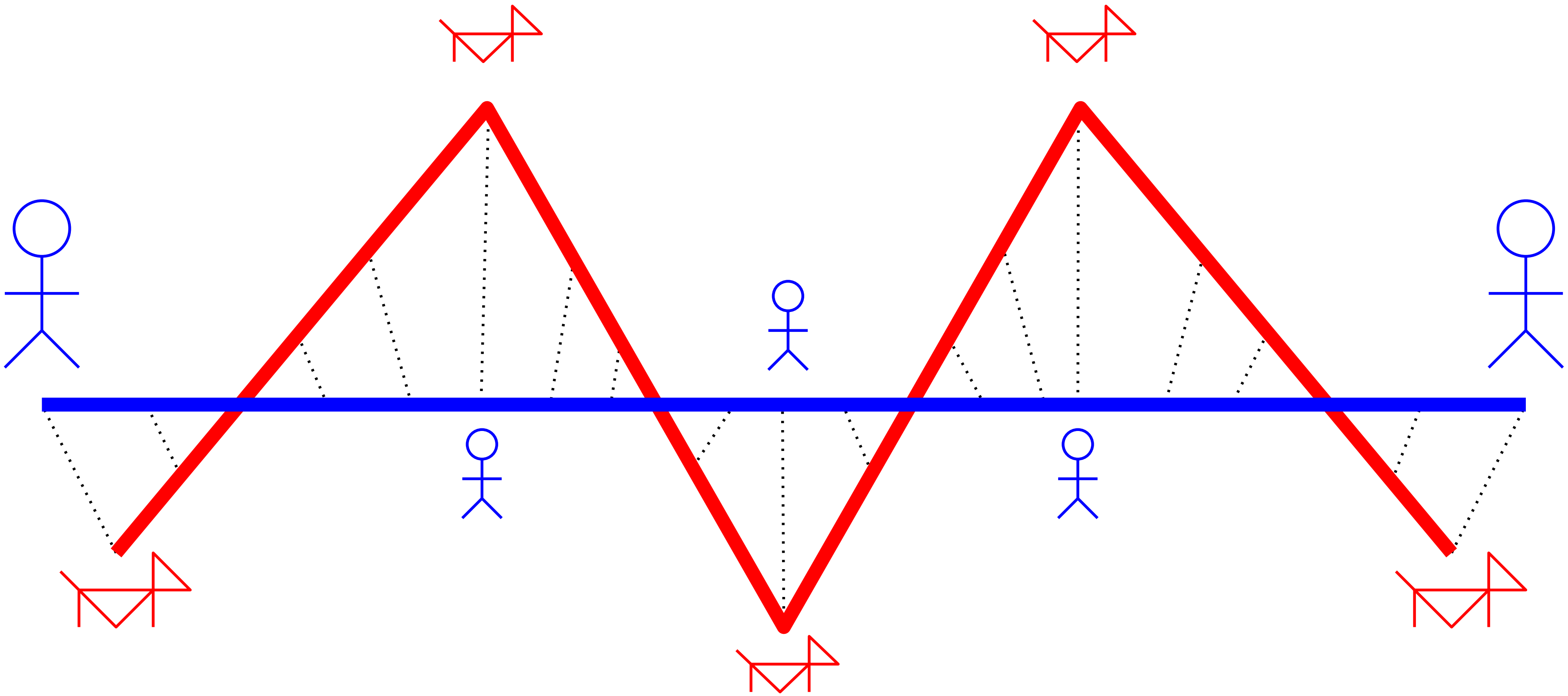
Fréchet distance between curves

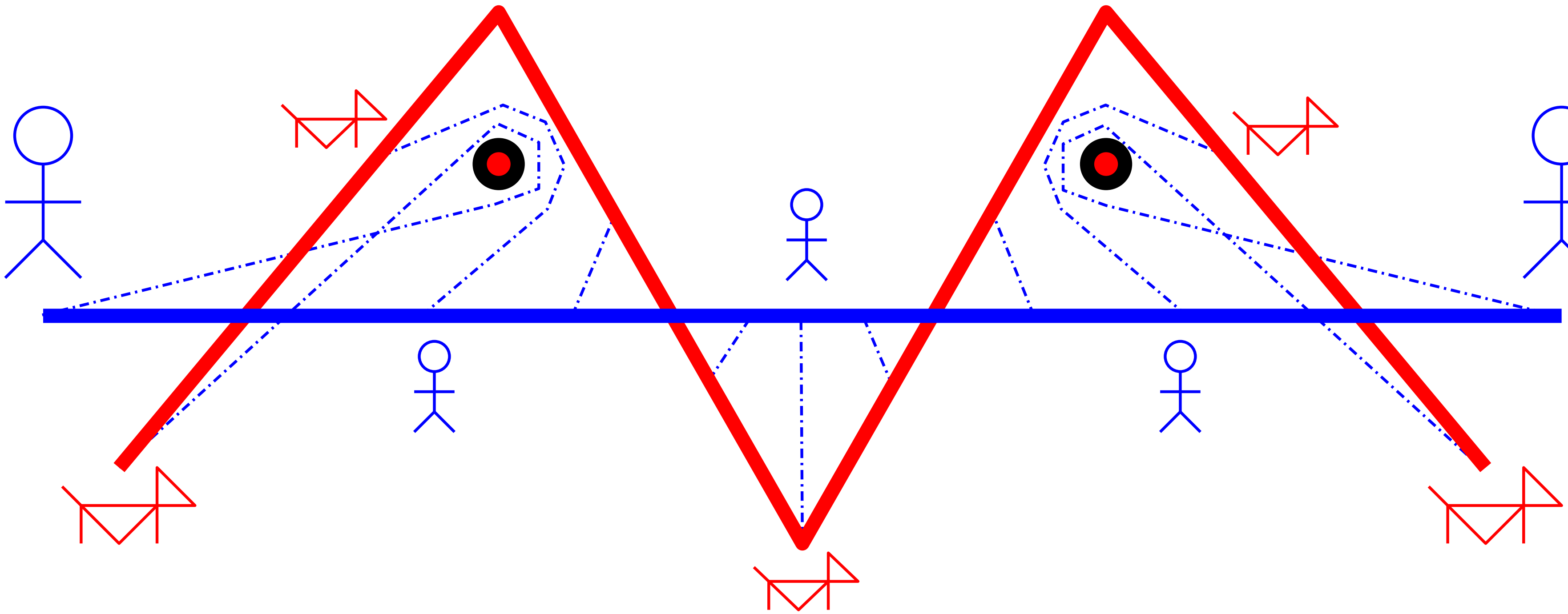


Dog-leash distance: Minimum length of a **straight** leash joining a dog and its owner that allows them to walk along their respective curves, from one endpoint to the other, continuously without backtracking

Woods have trees

. . . and other obstacles

New condition: Leash must move **continuously** in the ambient metric space

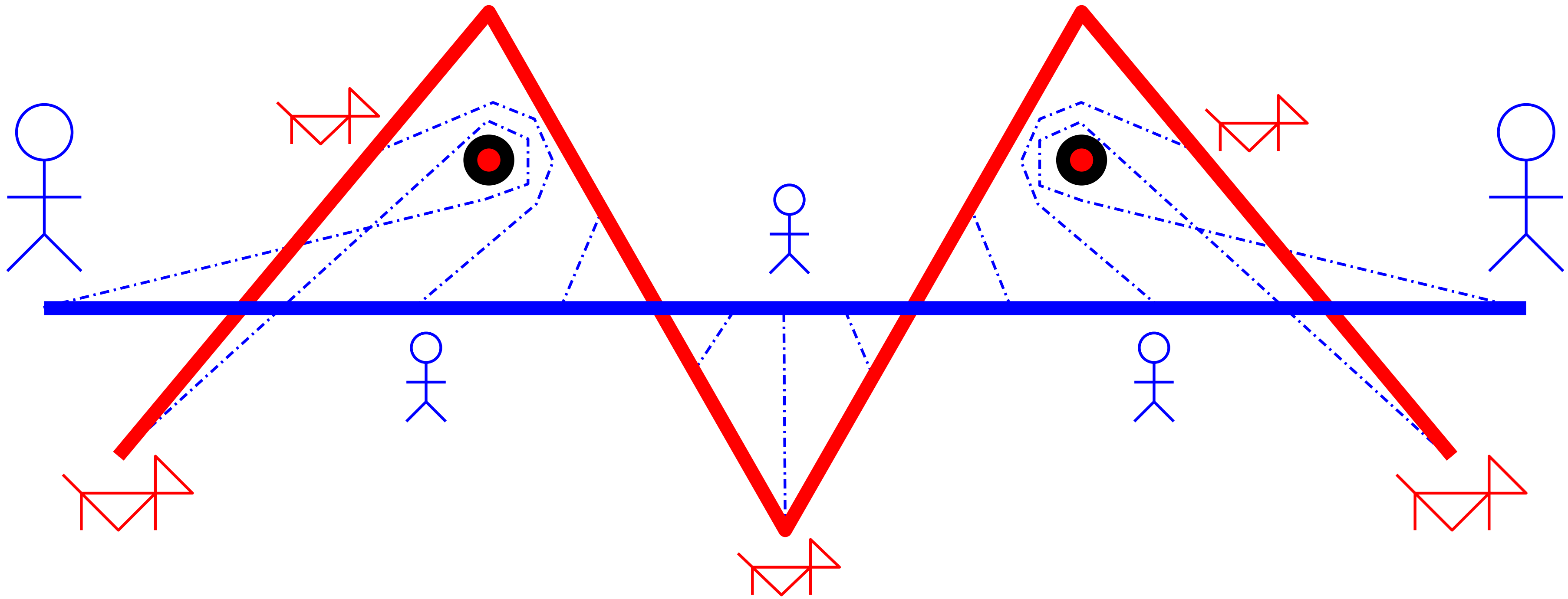


If there are obstacles, a longer leash may be required because the leash cannot jump over them

Goal: Walk the dog with the shortest leash possible

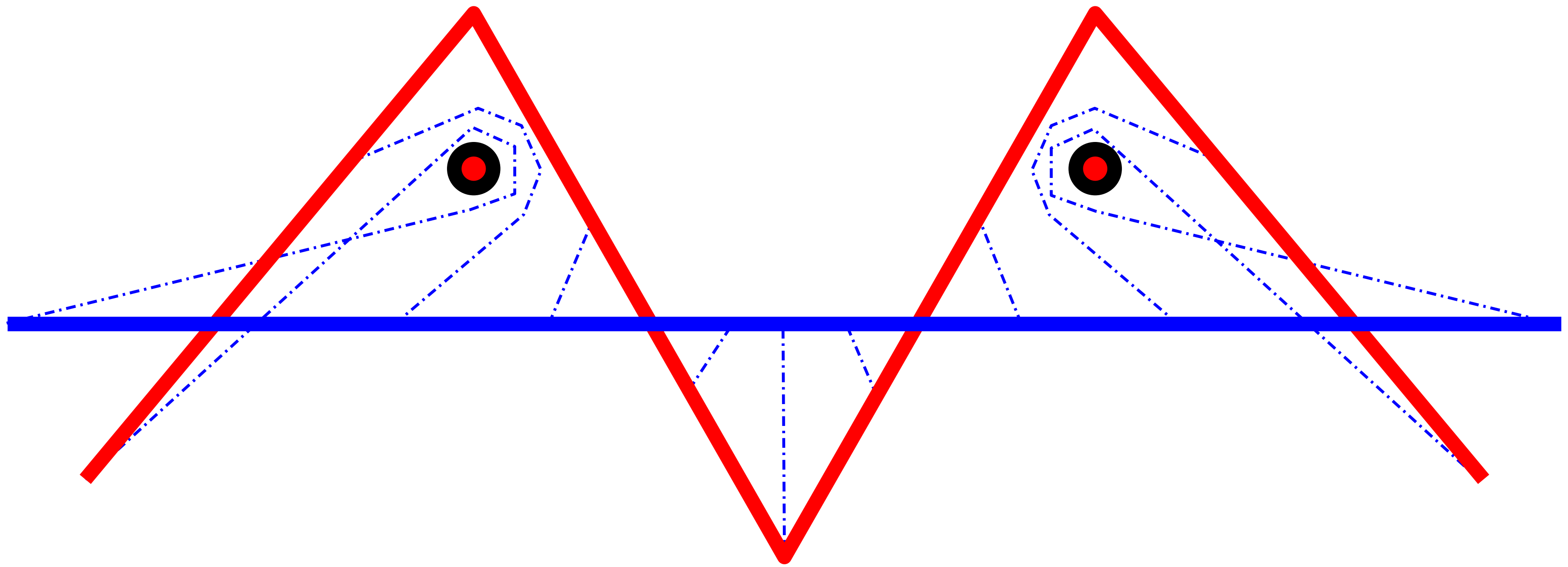
Homotopic Fréchet distance

Dog-leash distance in a general metric space where the leash must move continuously



We give a polynomial-time algorithm to compute the homotopic Fréchet distance between two polygonal curves in the plane with **polygonal obstacles**

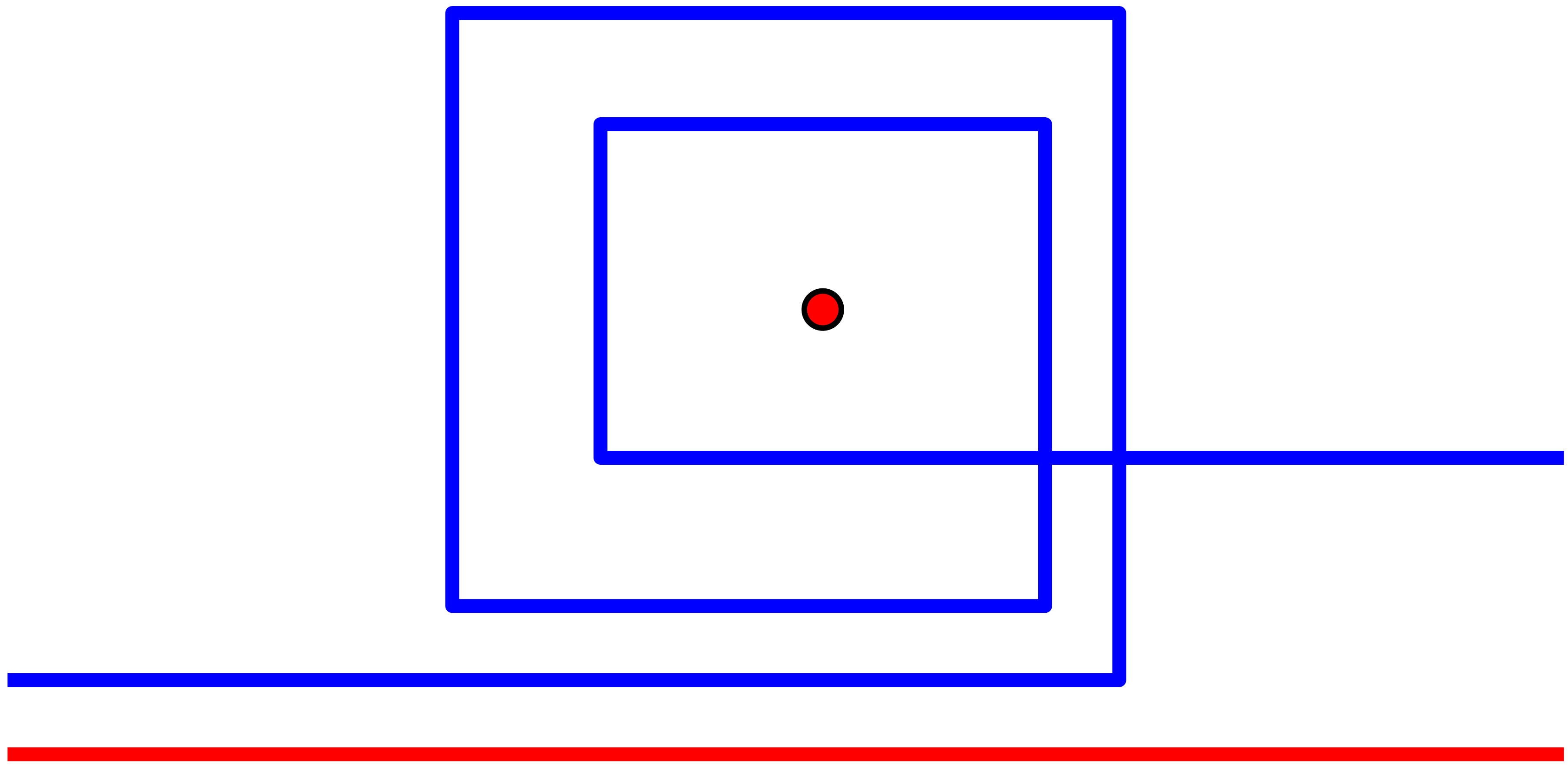
Application 1: Morphing



Leash motion encodes a continuous deformation between A and B, without penetrating obstacles

The "cost" of the deformation is the maximum distance any point has to travel

Application 2: Measuring similarity



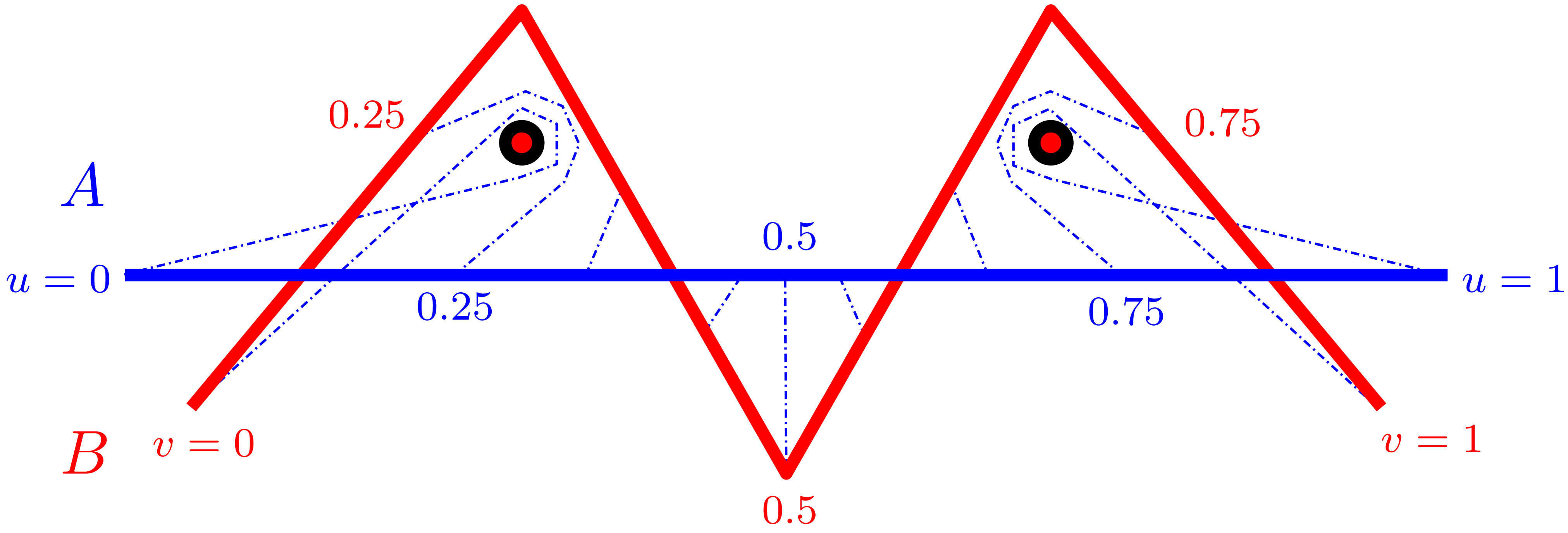
Robotics: Given two paths, how *similar* are they?

Paths that go around obstacles differently are not similar

Leash map

Continuous function $\ell : [0,1] \times [0,1] \to S$

arc-length time metric space



s.t. $\ell(\cdot, t)$ is the leash at time t joining $A(u(t))$ and $B(v(t))$
(ℓ encodes re-parameterizations u, v of A, B)

Homotopic Fréchet distance

The **cost** of a leash map ℓ is the longest length of the leash at any time during the leash motion:

$$\text{cost}(\ell) := \sup_{t \in [0,1]} \{\, \text{Length of } \ell(\cdot, t) \,\}$$

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The **homotopic Fréchet distance** is the minimum cost of any leash map:

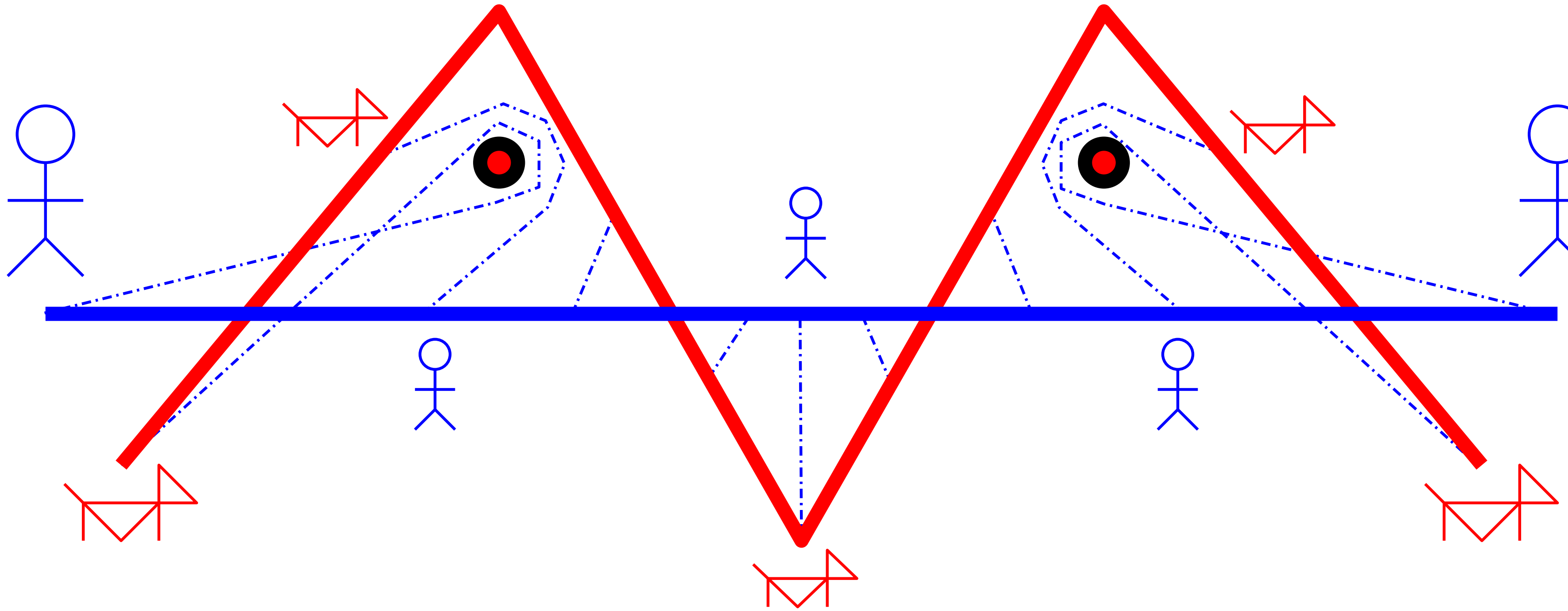
$$F(A, B) := \inf_{\text{leash map } \ell} \{\, \text{cost}(\ell) \,\}$$

Punctured plane

Let A, B be two given curves in $\mathbb{E}^2$

Let P be a set of points in the plane $=$ obstacles

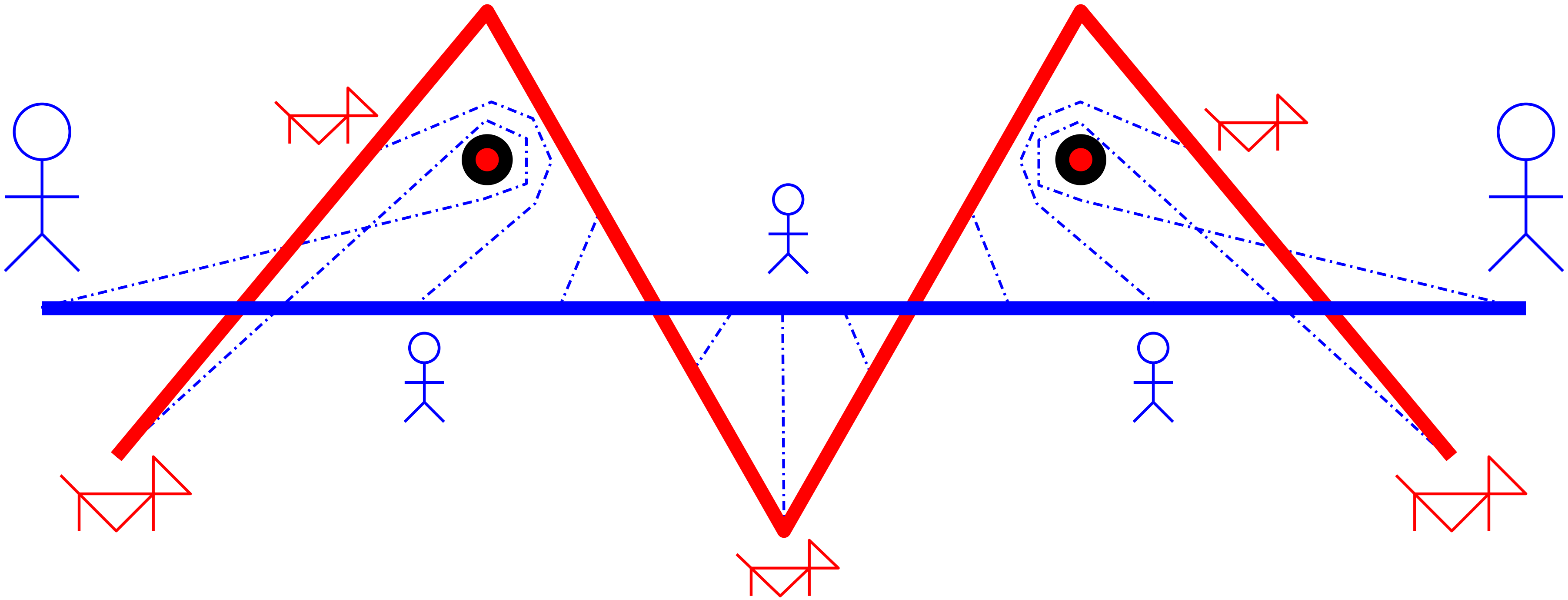
A **leash** is a curve joining a point of A and a point of B



Leash must move continuously in the **punctured plane** $\mathbb{E}^2 \setminus P$, so it cannot jump over obstacles

Relative homotopy

Two leashes are **relatively homotopic** if one can be continuously transformed into the other in the punctured plane *while keeping their endpoints on the respective curves*



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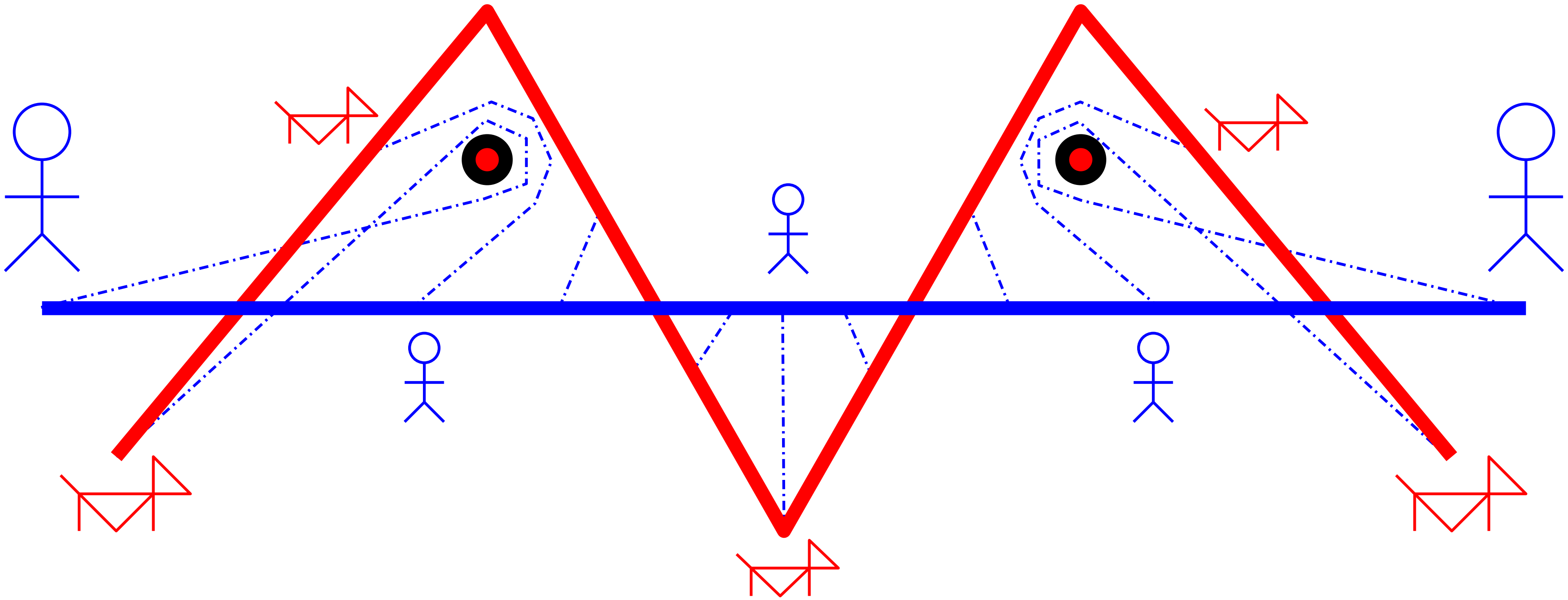

Every leash map ℓ_h describes a set of leashes belonging to some relative homotopy class h

Homotopic Fréchet distance redux

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$$\text{Define } \; F_h(A, B) := \inf_{\ell_h} \{ \operatorname{cost}(\ell_h) \}$$

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Homotopic Fréchet distance

$$F(A,B) := \min_h \{F_h(A,B)\}$$

Geodesic leashes

Lemma: There exists an optimum leash map such that the leash at every time is the shortest path in its homotopy class

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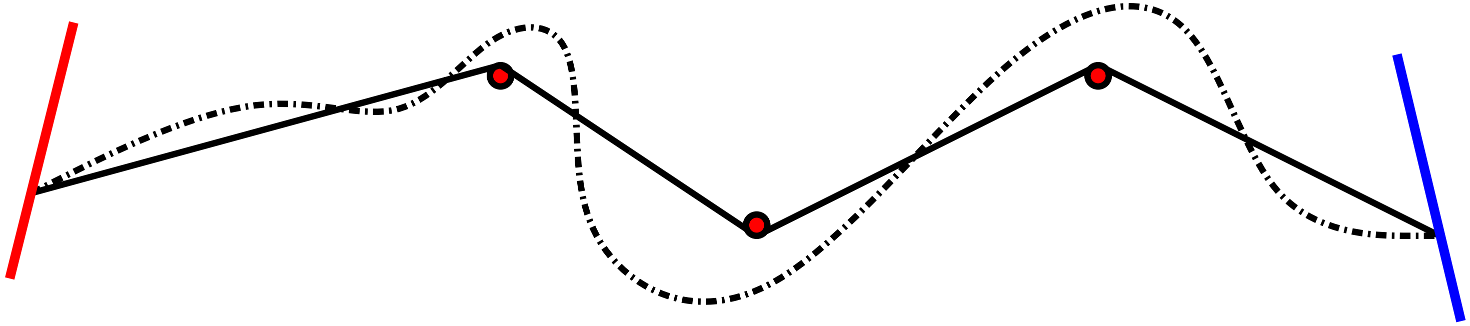

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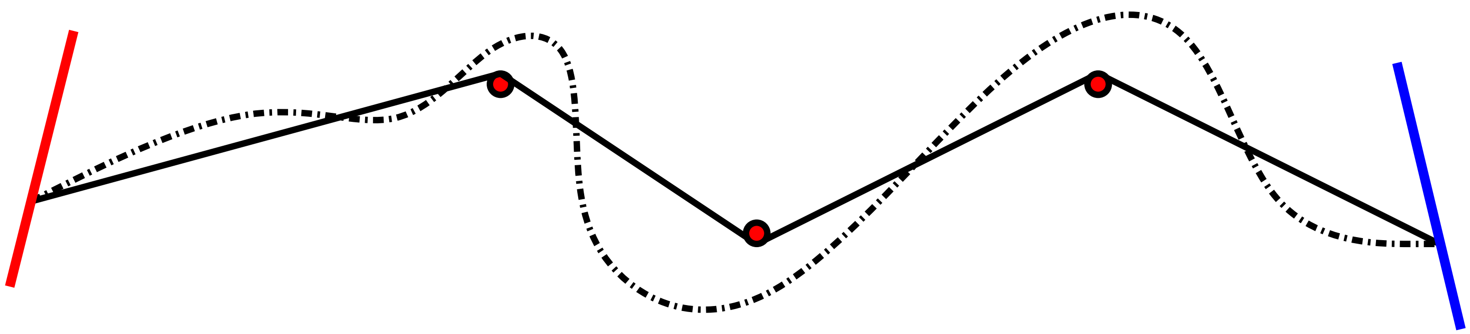
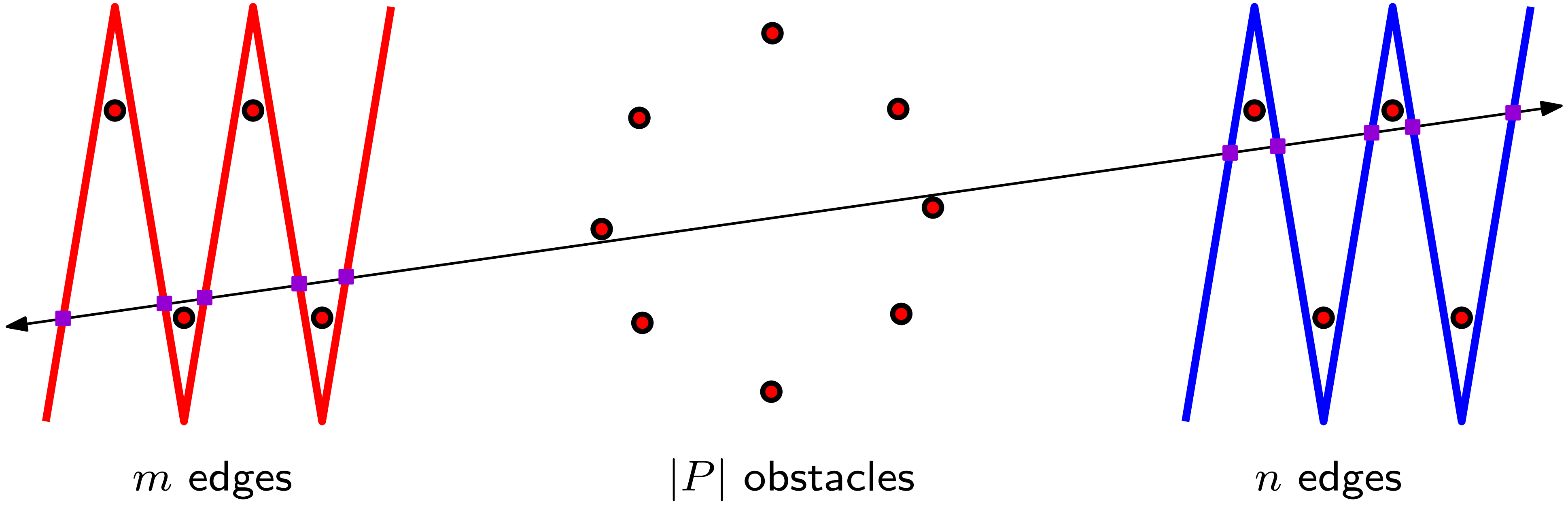


We allow the leash to pass through obstacle points. A turning angle at every obstacle point uniquely identifies the homotopy class of the leash. Now, unique shortest paths exist in every homotopy class.

Key lemma

The optimum homotopy class h^* must contain a straight-line leash



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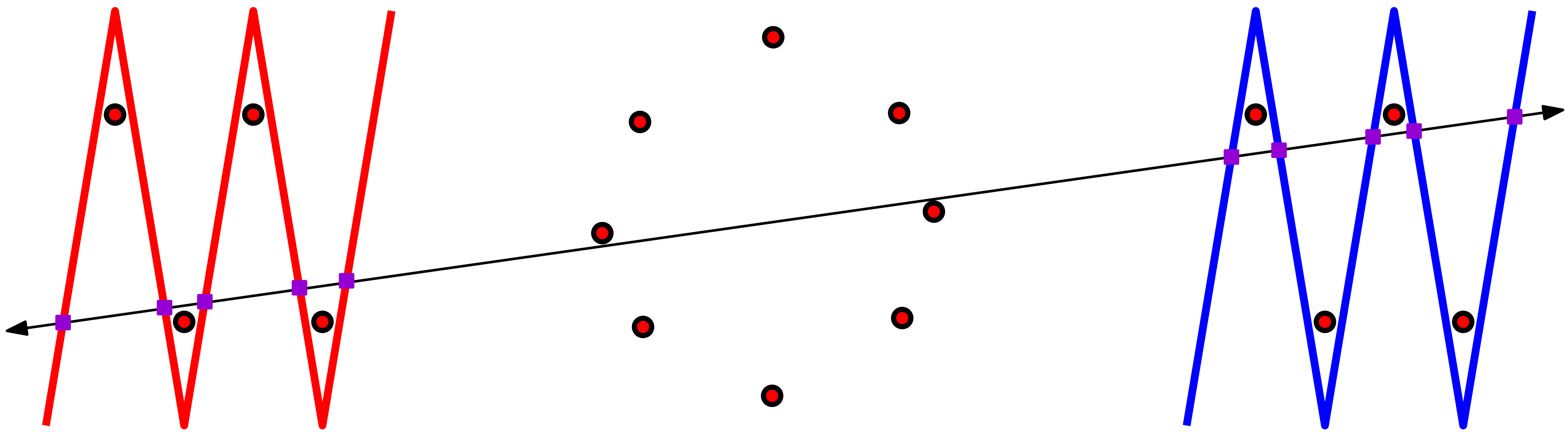
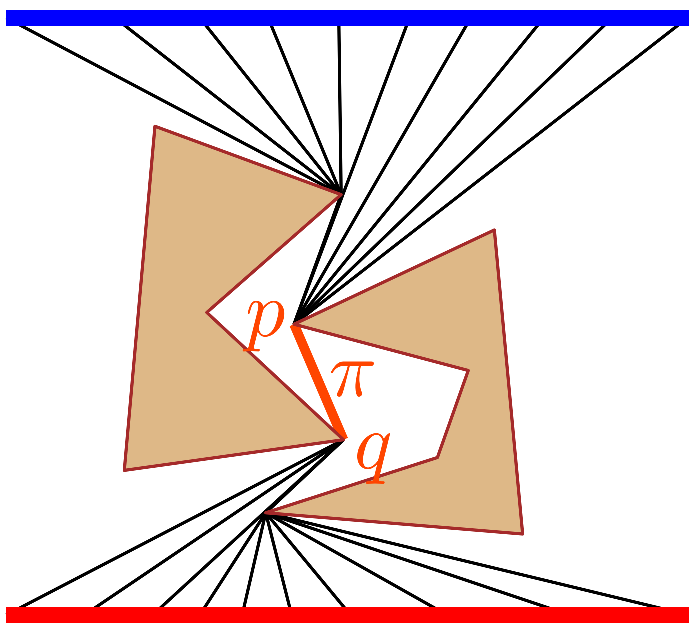


Our algorithm: List all $O(mn|P|^2)$ homotopy classes h that contain a straight-line leash and compute $F_h(A, B)$ in $O(mn|P| \log mn|P|)$ time using **parametric search**

Polygonal obstacles

The optimum leash map ℓ^* may be **pinned** at a common subpath π, i.e., a globally shortest p-q path



Enumerate $O(mn|P|^4)$ homotopy classes h

Compute $F_h(A, B)$ from two independent leash maps

Open: On a convex polyhedron

Leash is not always a geodesic!

e.g., leash must have enough slack to cross over a vertex (a 'mountain')

Challenge: Characterize an optimum leash map

Thank you!

Extra slides

Computing F_h

Decision problem: Given a real $d \geq 0$, is $F_h(A, B) \leq d$?

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$$d_1 \leq d_2 \leq \ldots \leq d_{i-1} \leq d^* \leq d_{i+1} \leq d_{i+2} \leq \ldots$$

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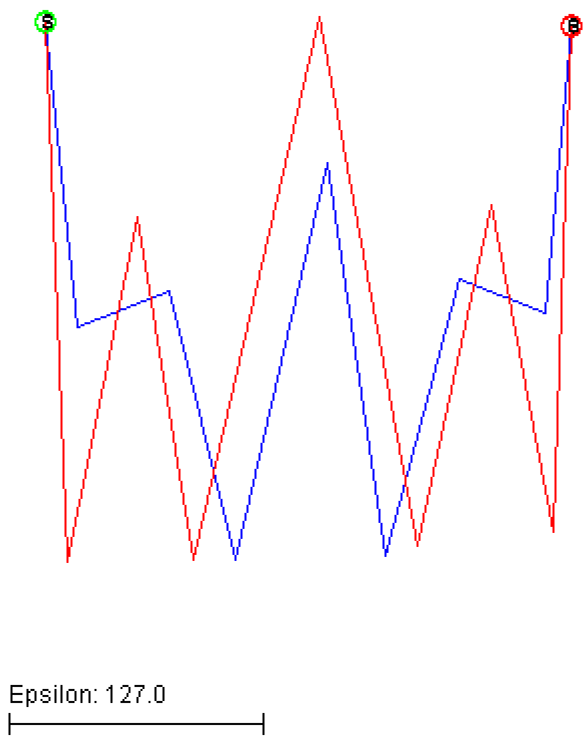
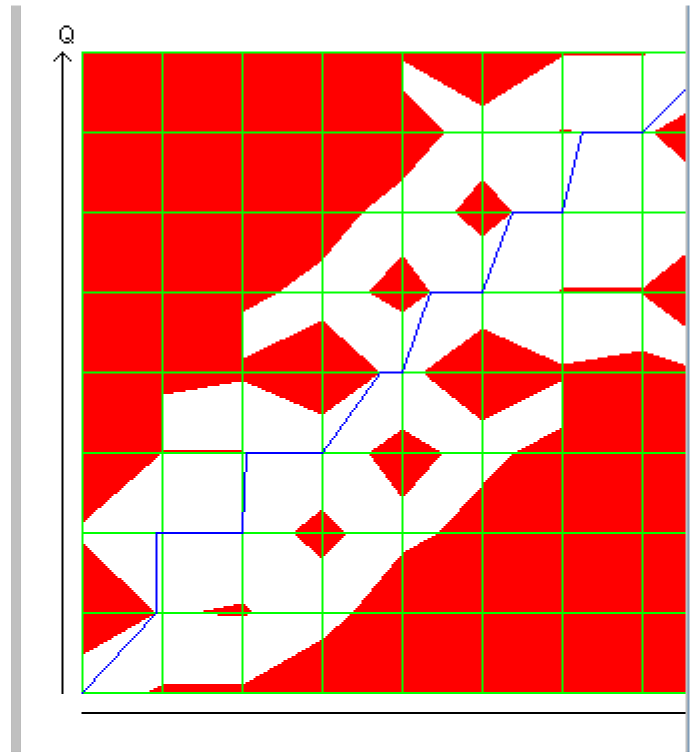
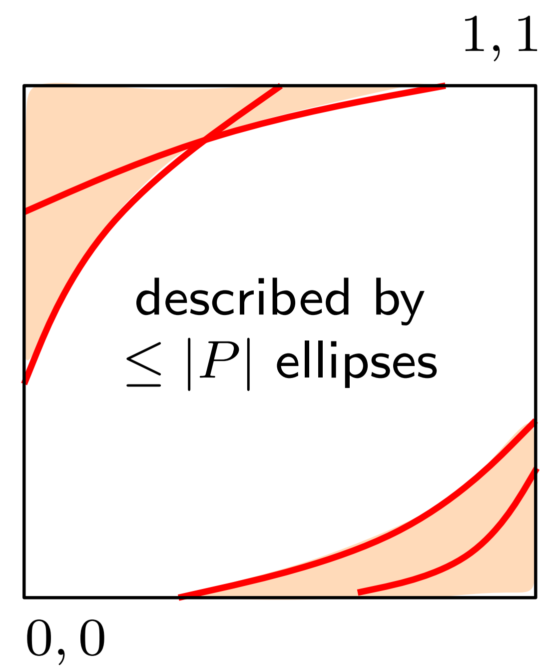
$d^* = F_h(A, B)$

Goal: Find the smallest critical value d for which the answer above is ‘yes’

Is $F_h \leq d$?







http://www.cim.mcgill.ca/~stephane/cs507/Project.html, Stéphane Pelletier, 2002

Is there a monotone path from $(0,0)$ to (m,n) in free space?

Lemma: In each cell C_{ij}, the free space is convex

Parametric search [Megiddo '83]

Let A_s be an algorithm to decide, given a critical value d_i, whether $F_h(A,B) \leq d_i$, with running time $O(T_s)$

Ask me later if you want me to describe A_s

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Total running time $= O(T_s^2)$

Parametric search on steroids [M'83]

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Total running time $= O(T_s T_p \log k + kT_p)$ *better than $O(T_s^2)$*

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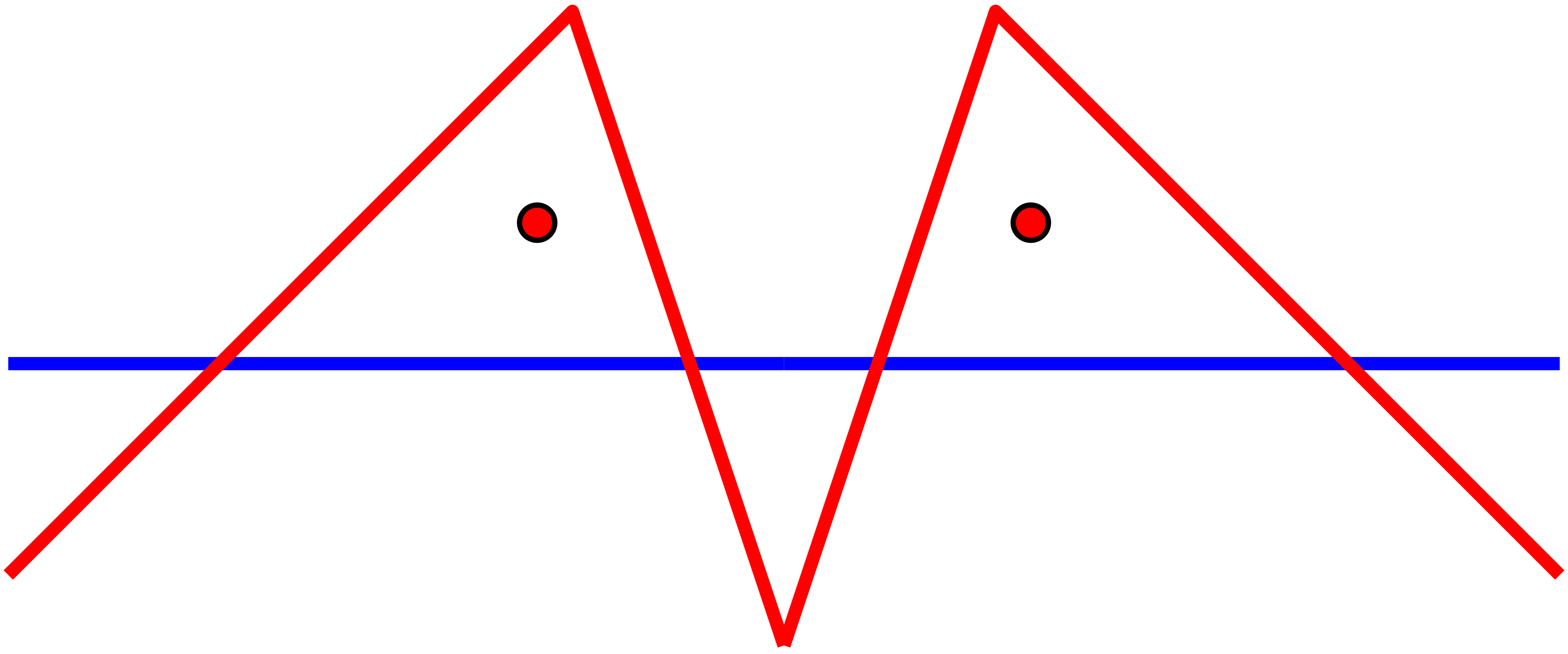

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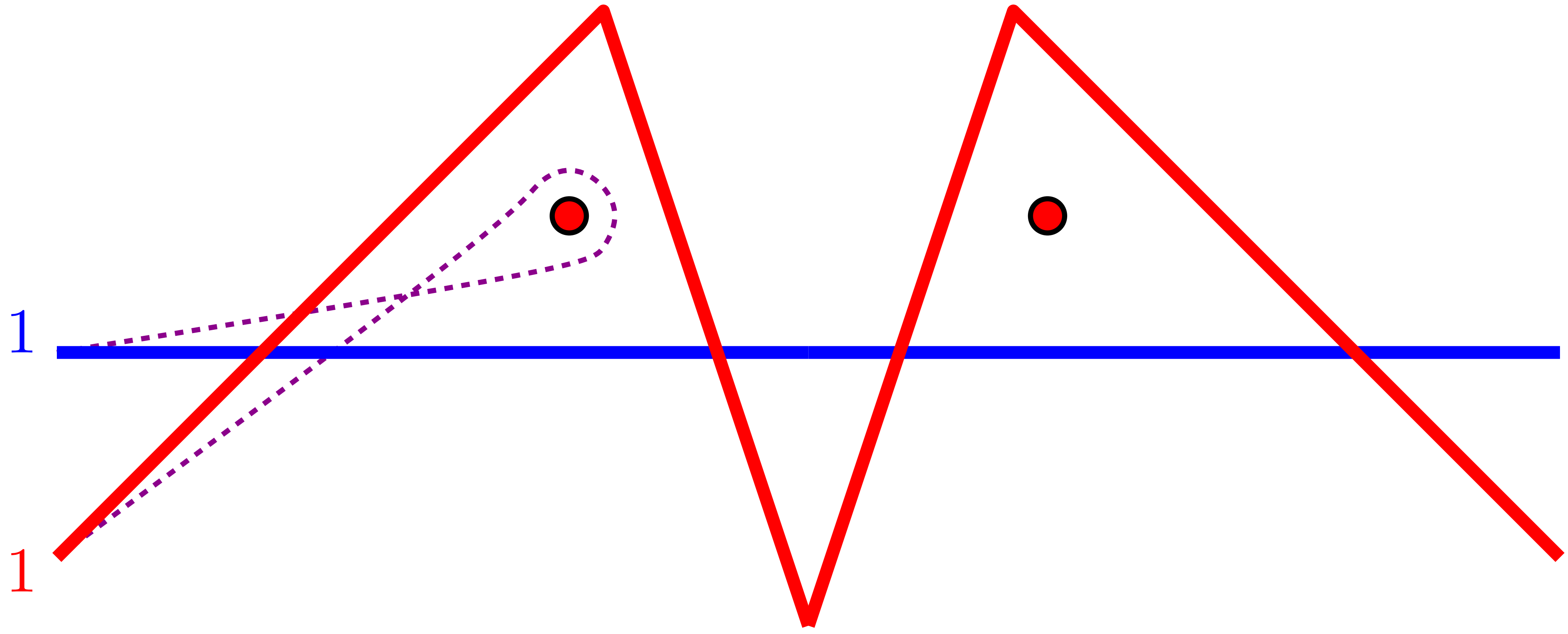

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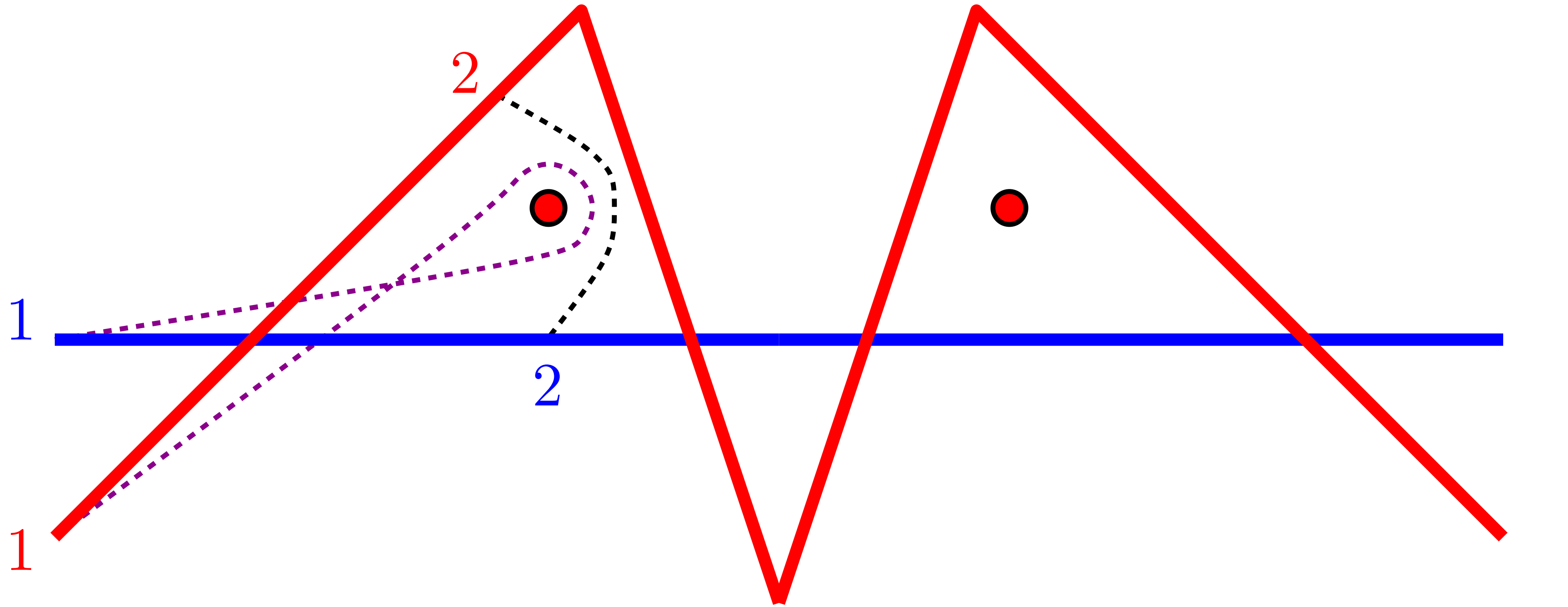

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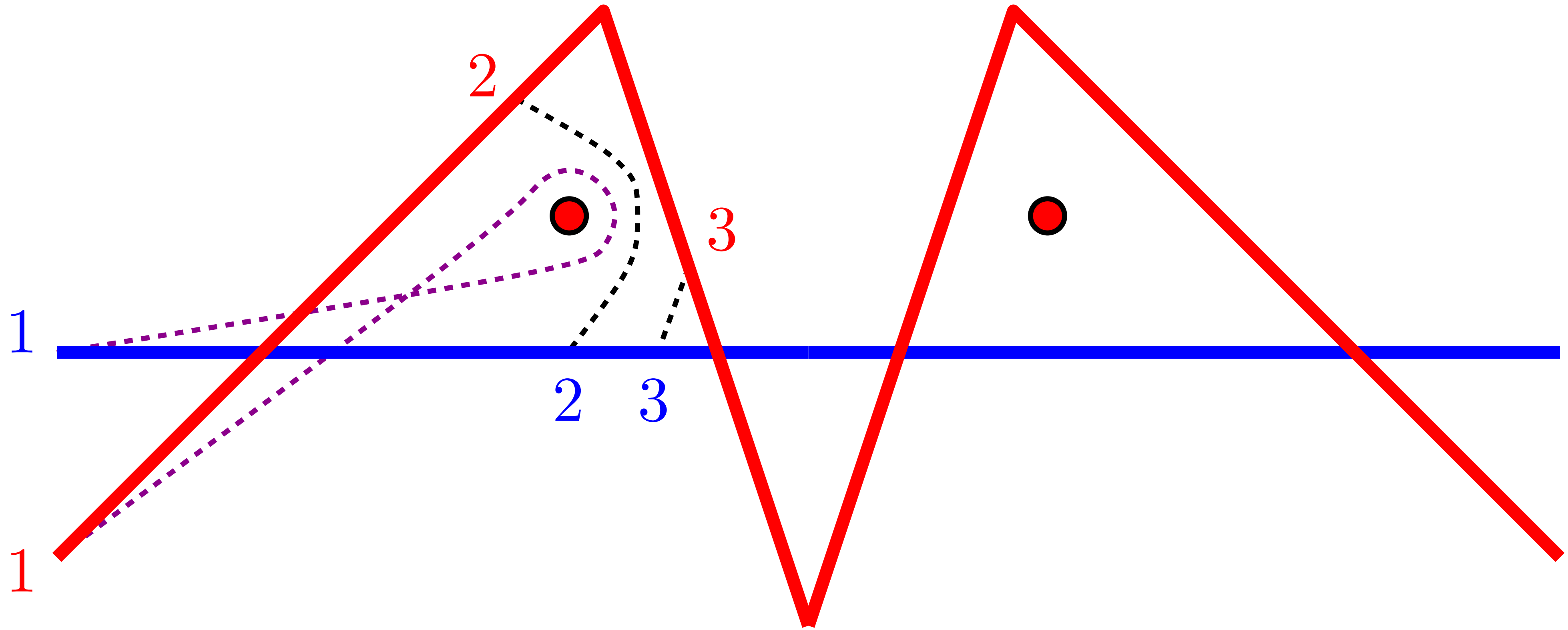

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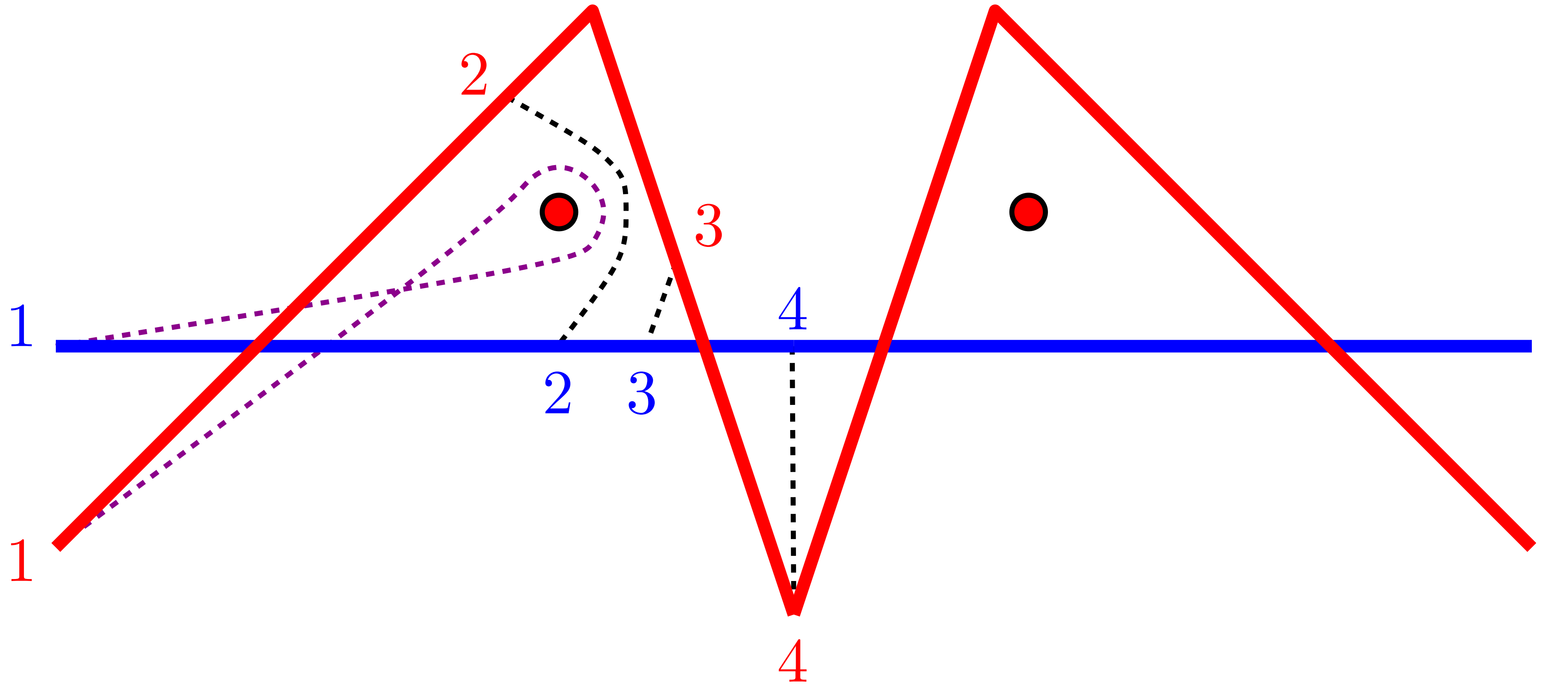

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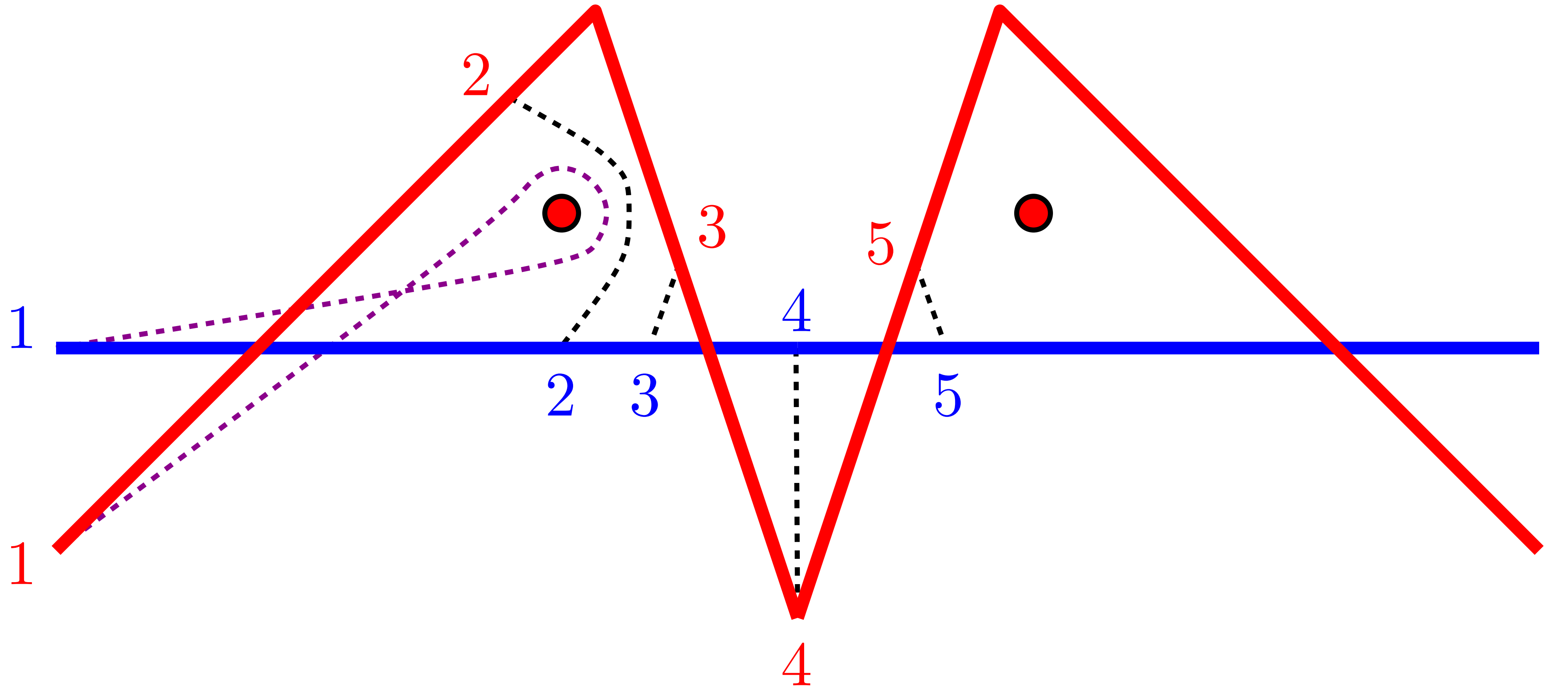

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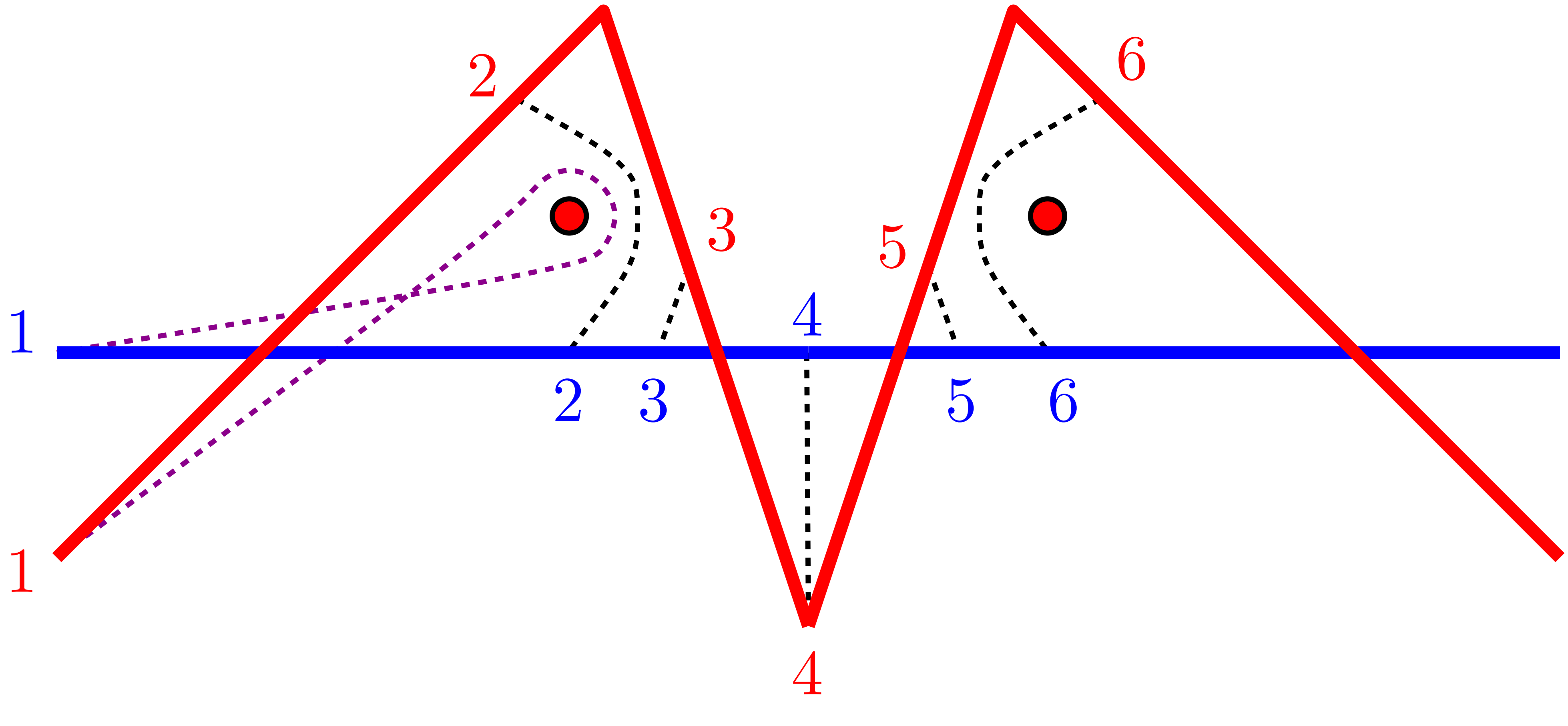

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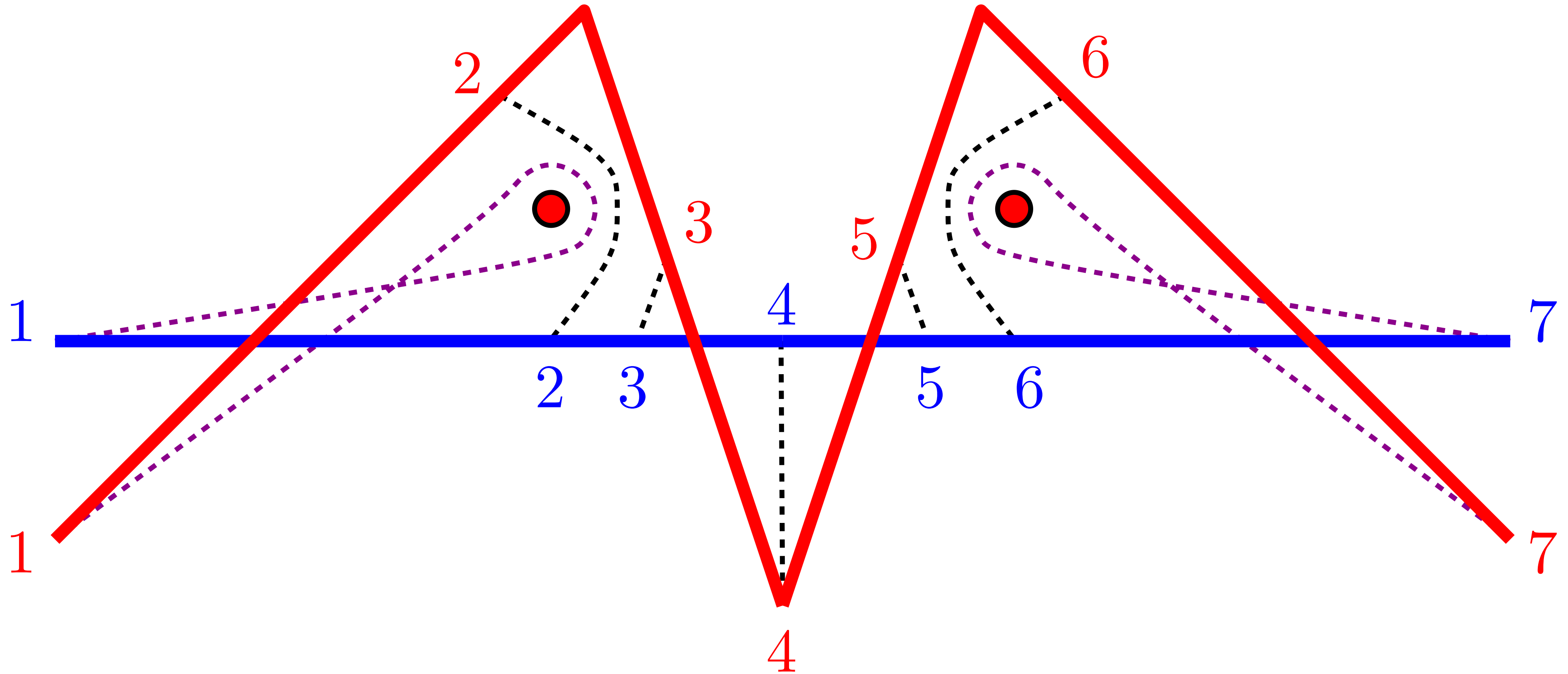

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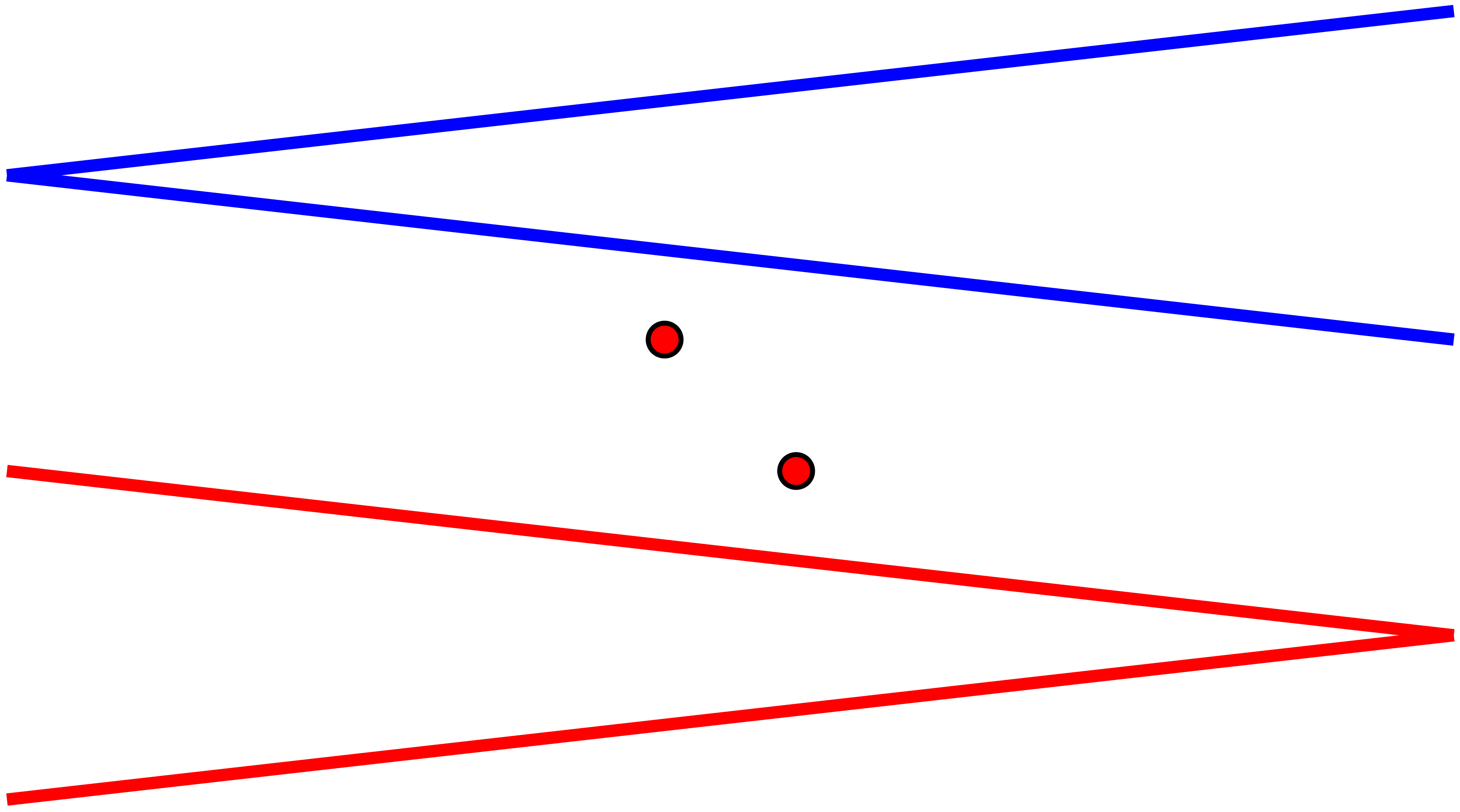

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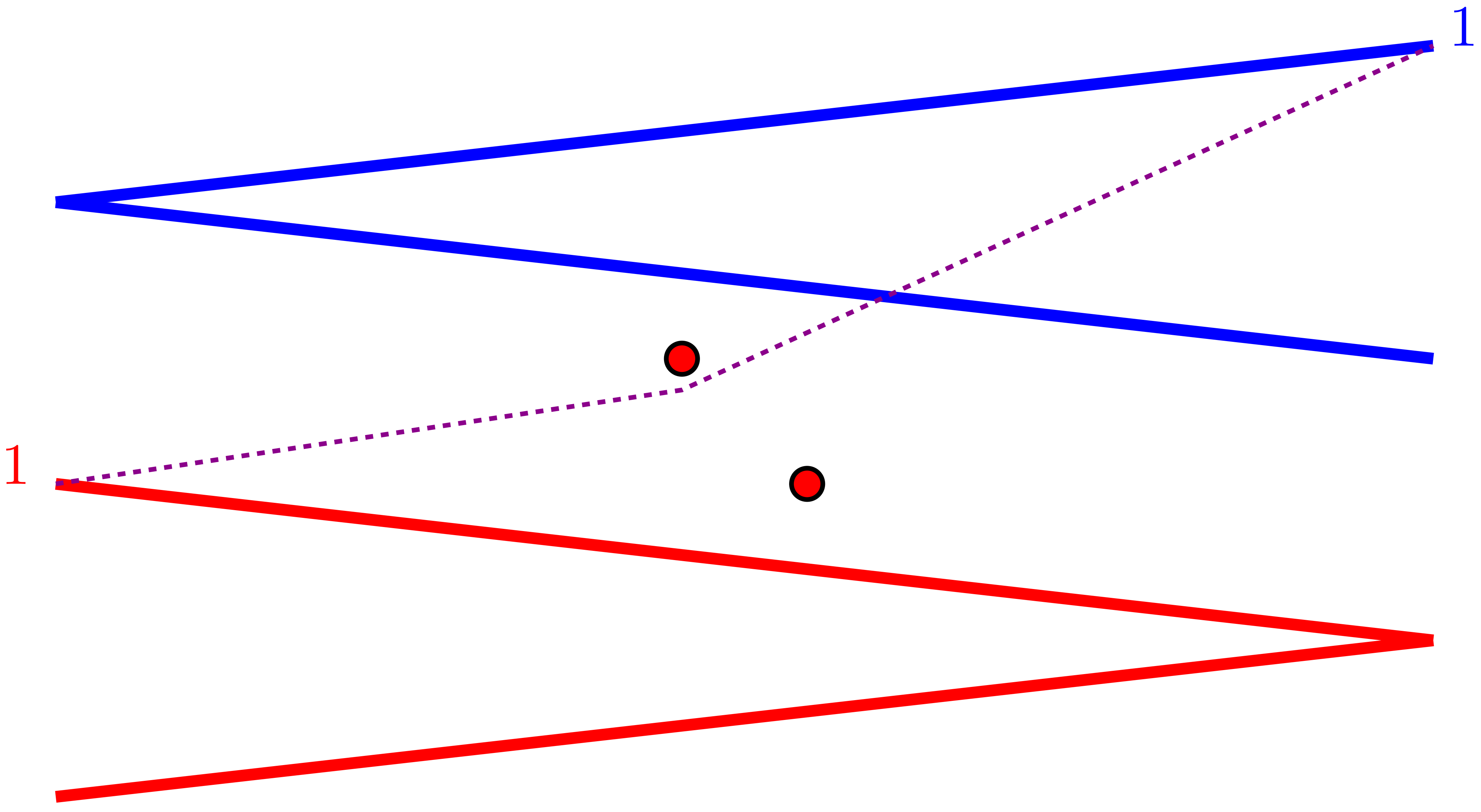

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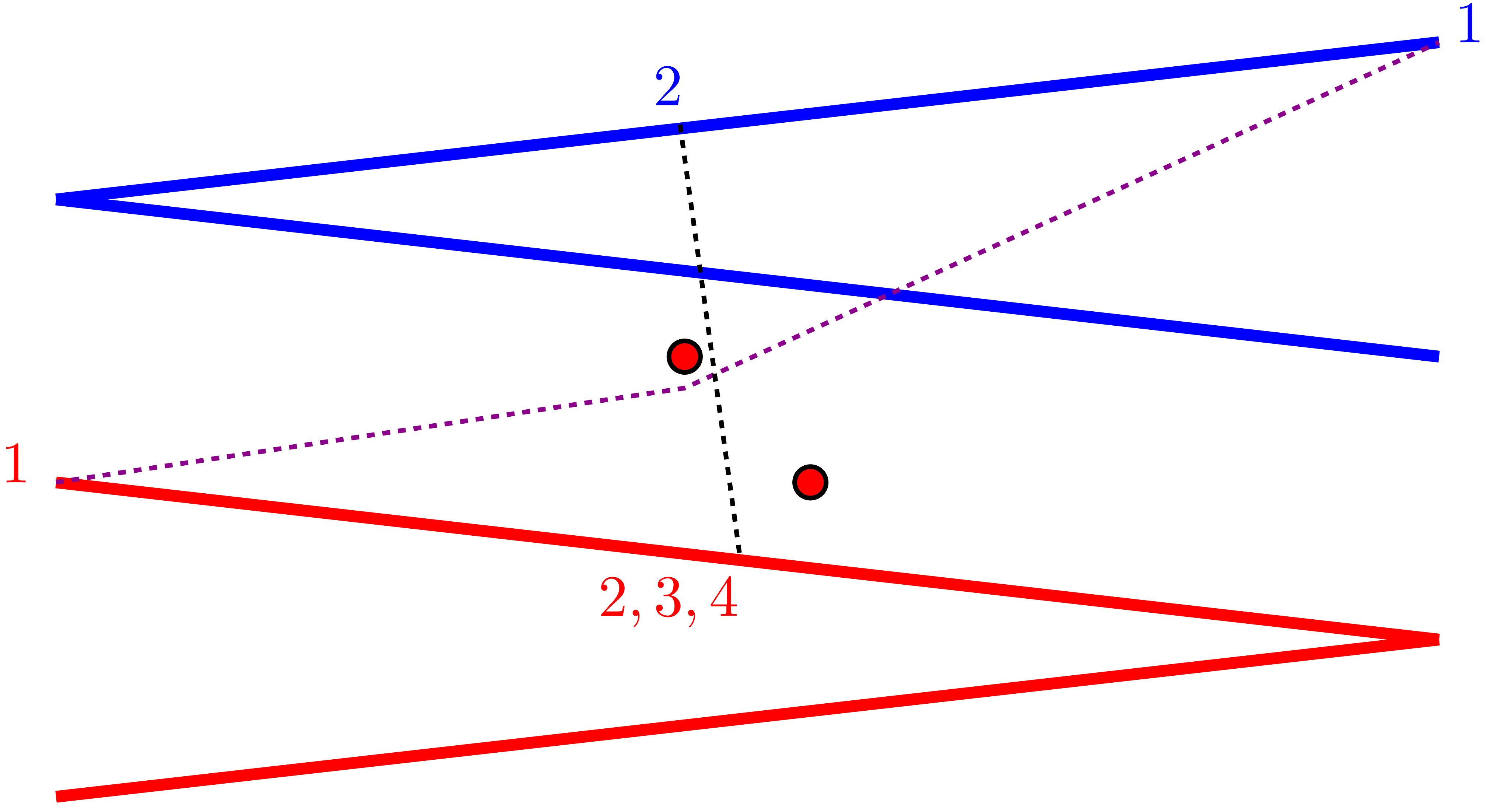

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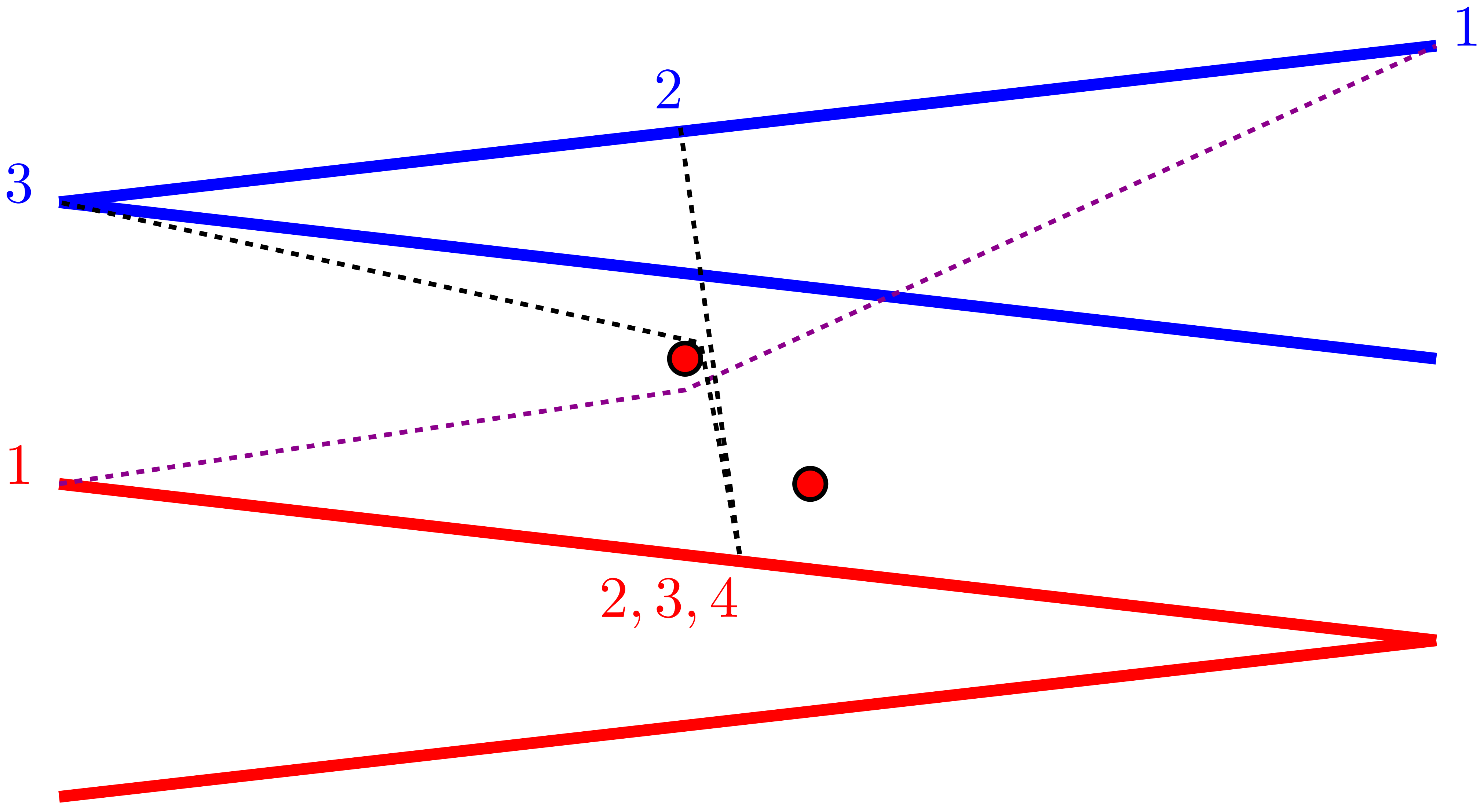

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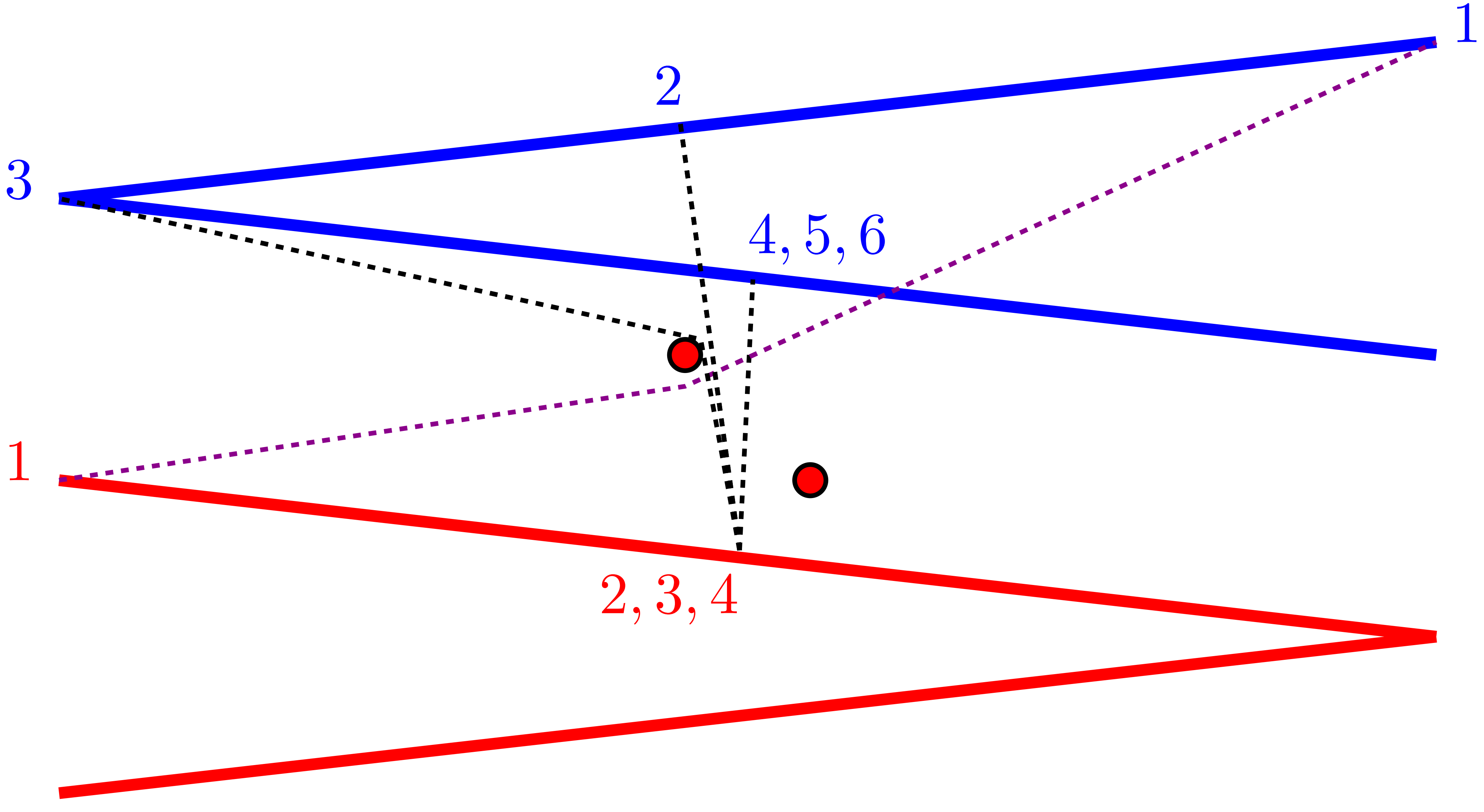

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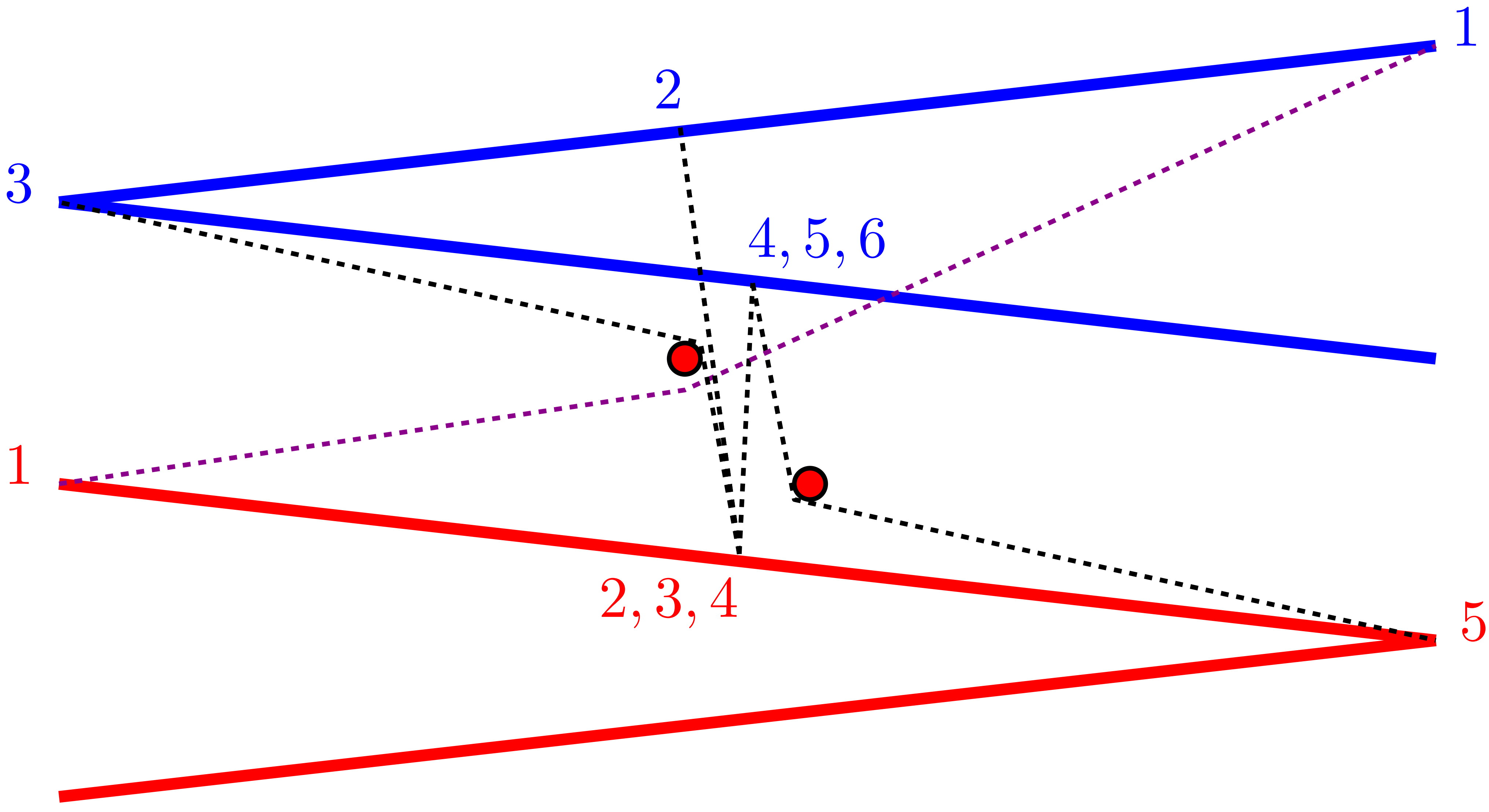

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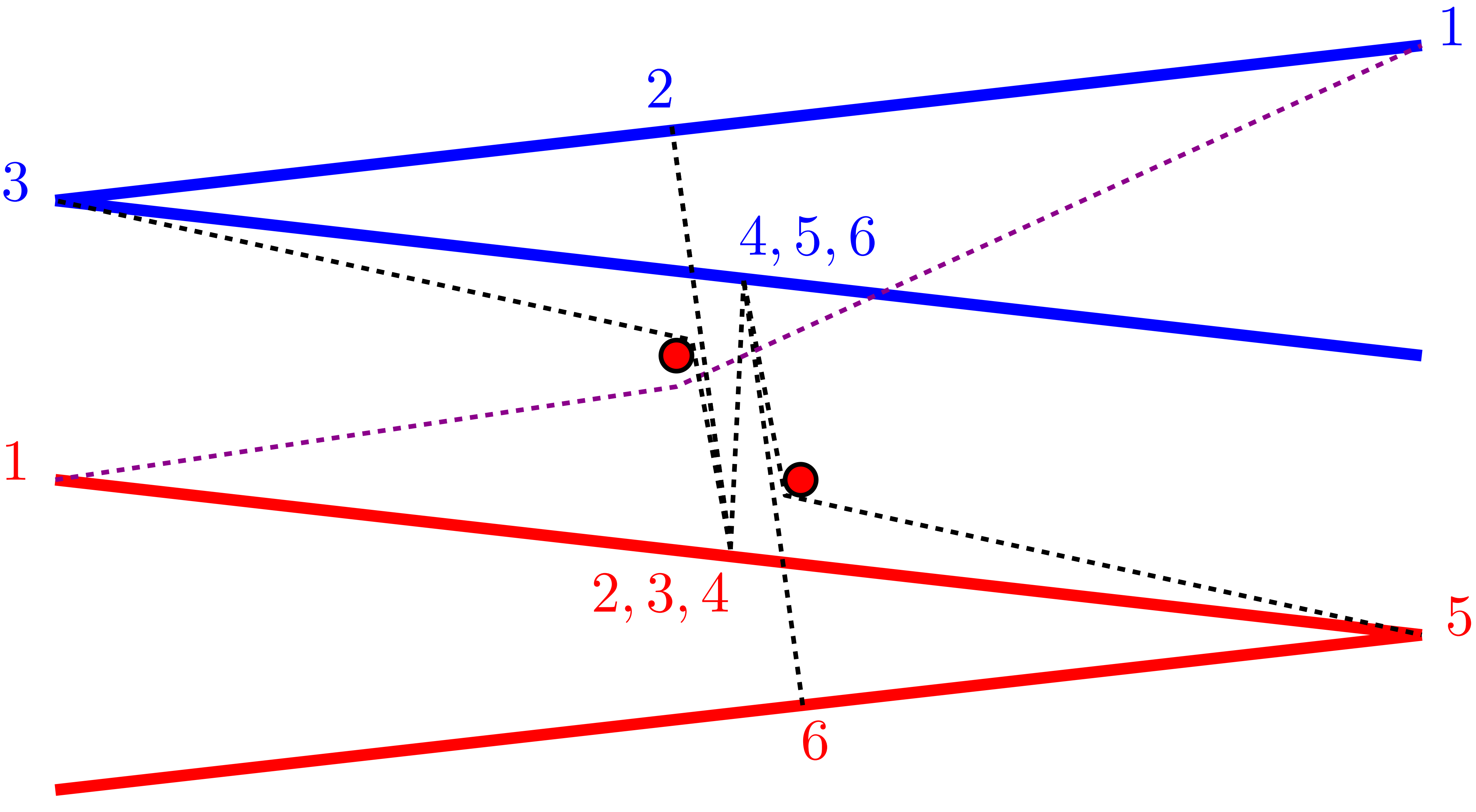

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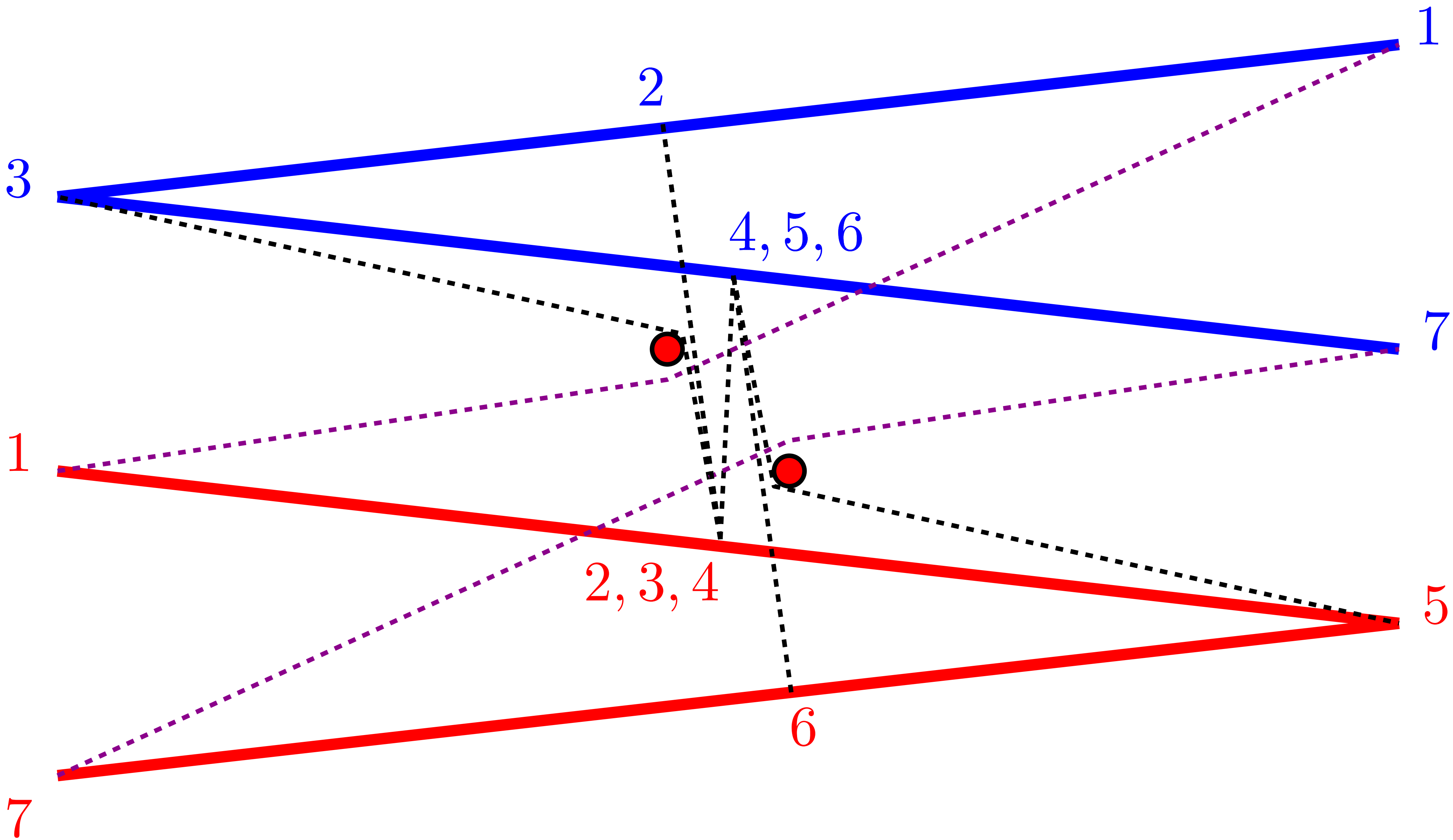

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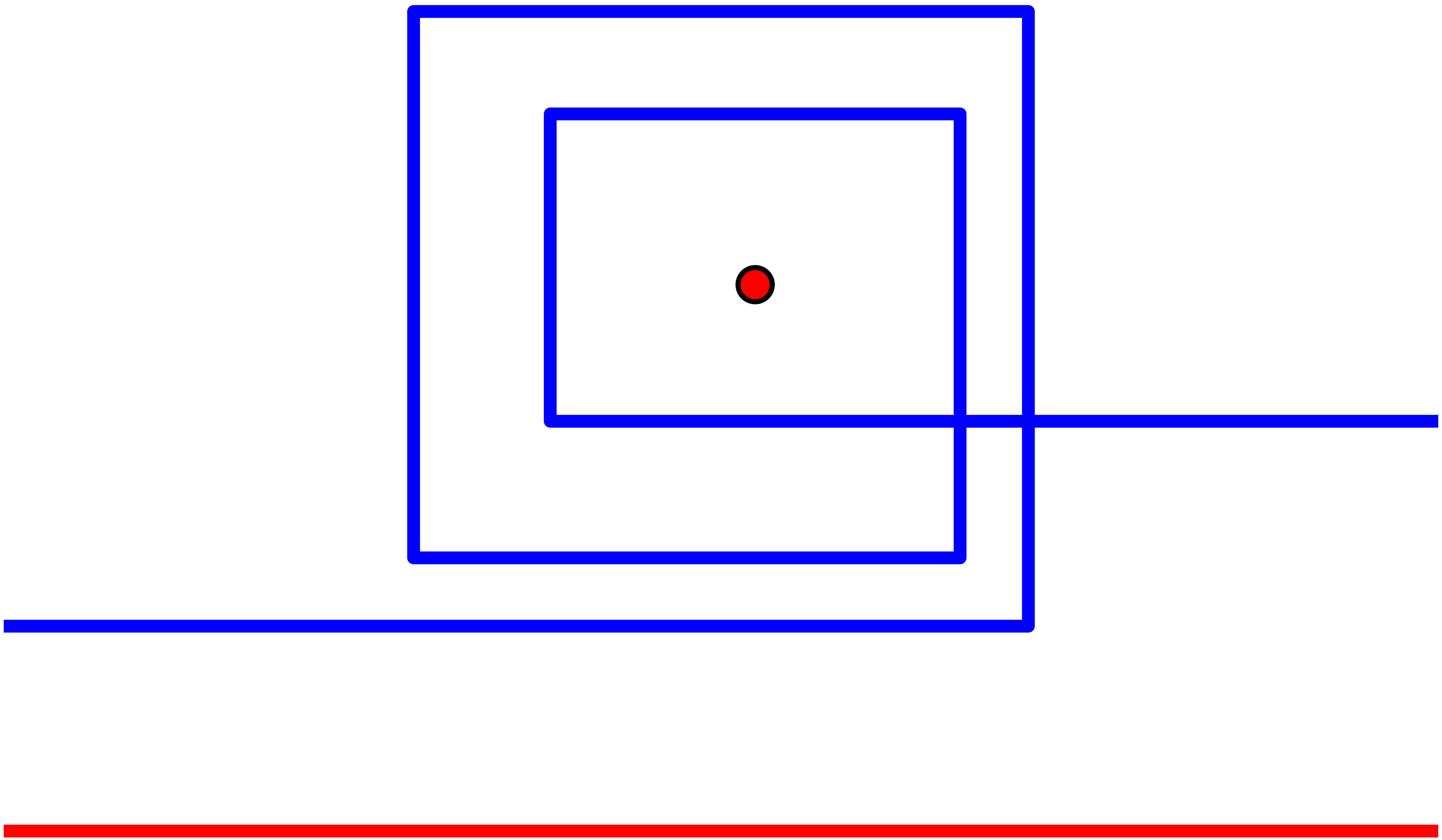

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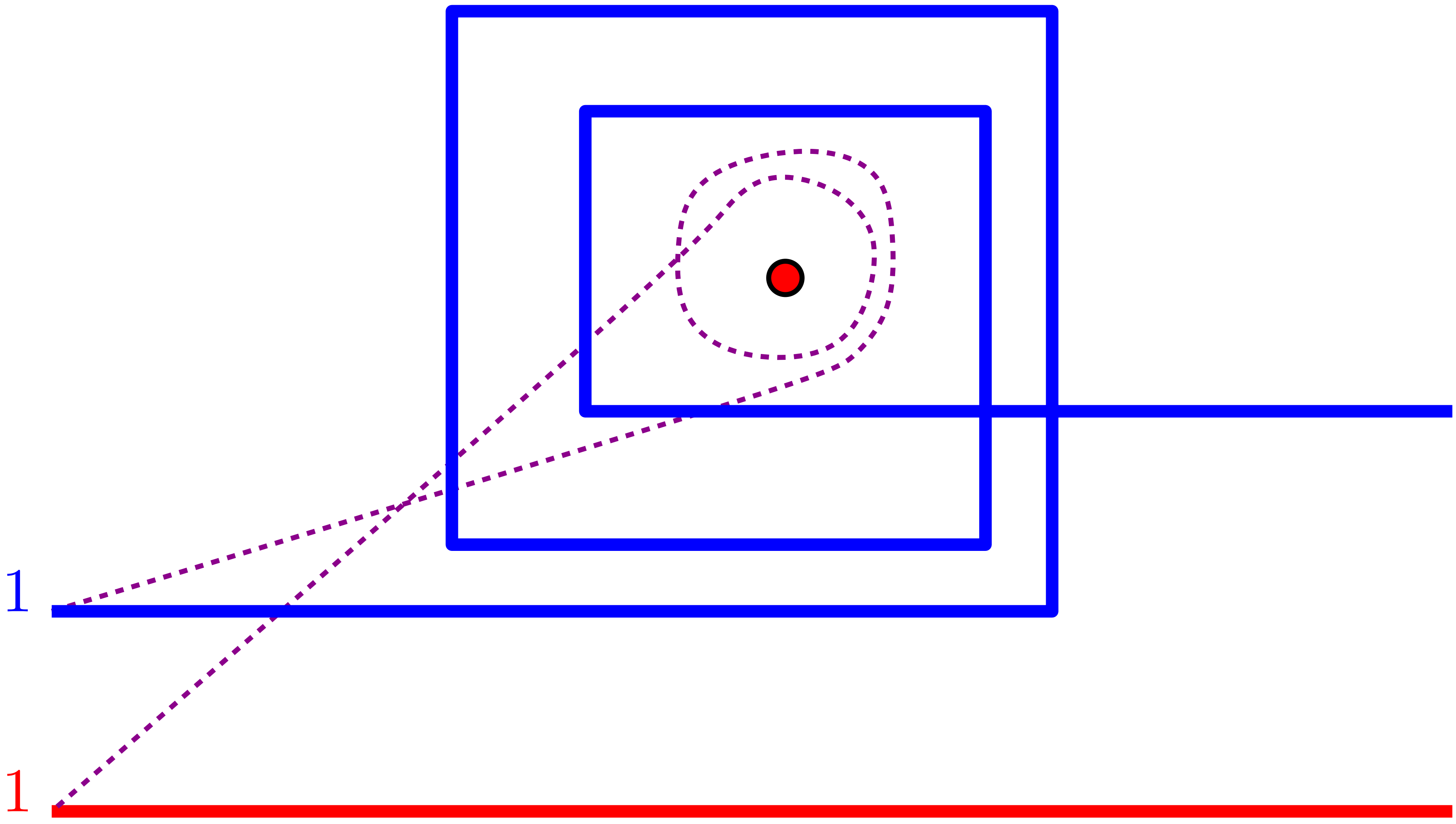

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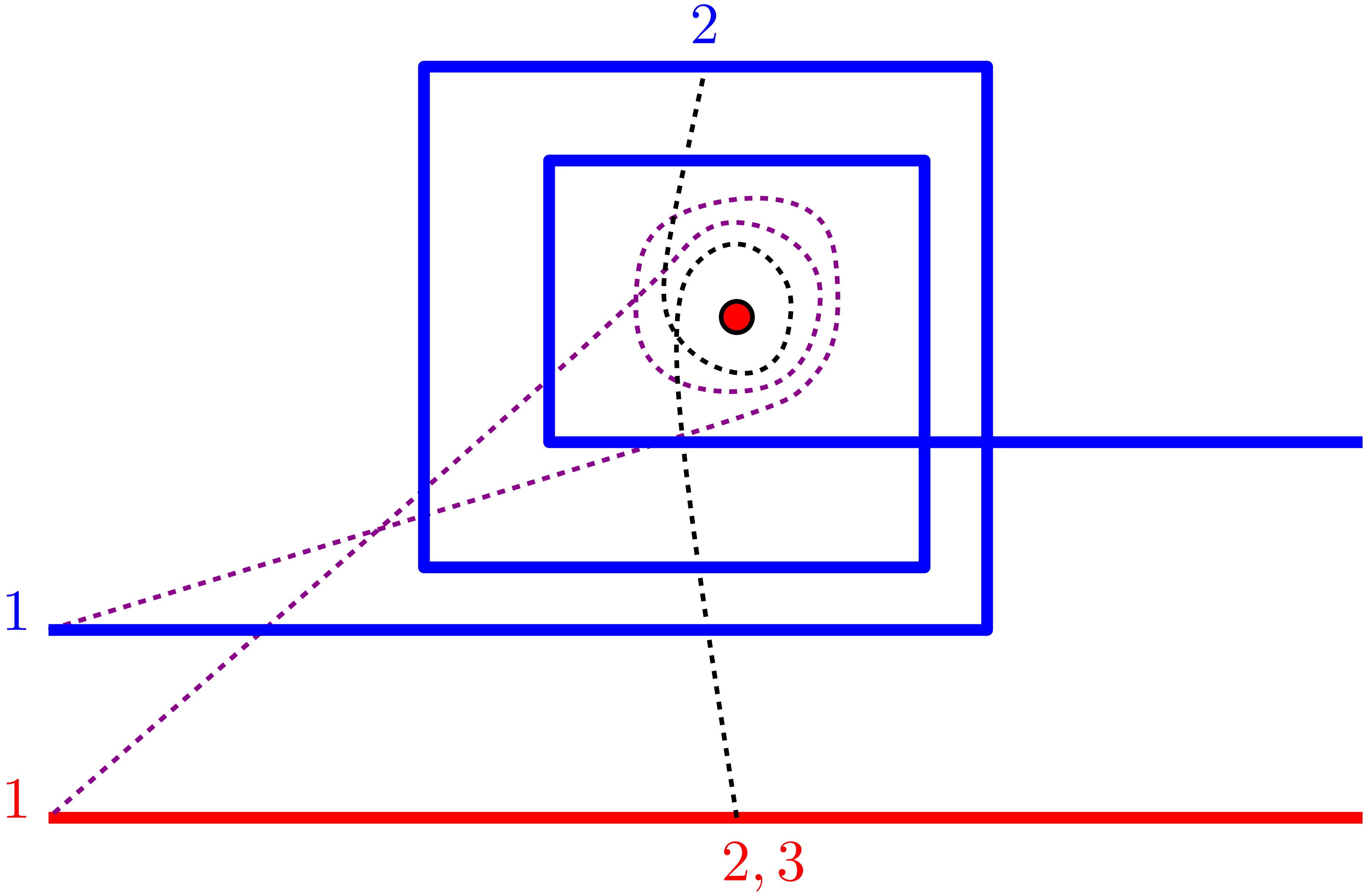

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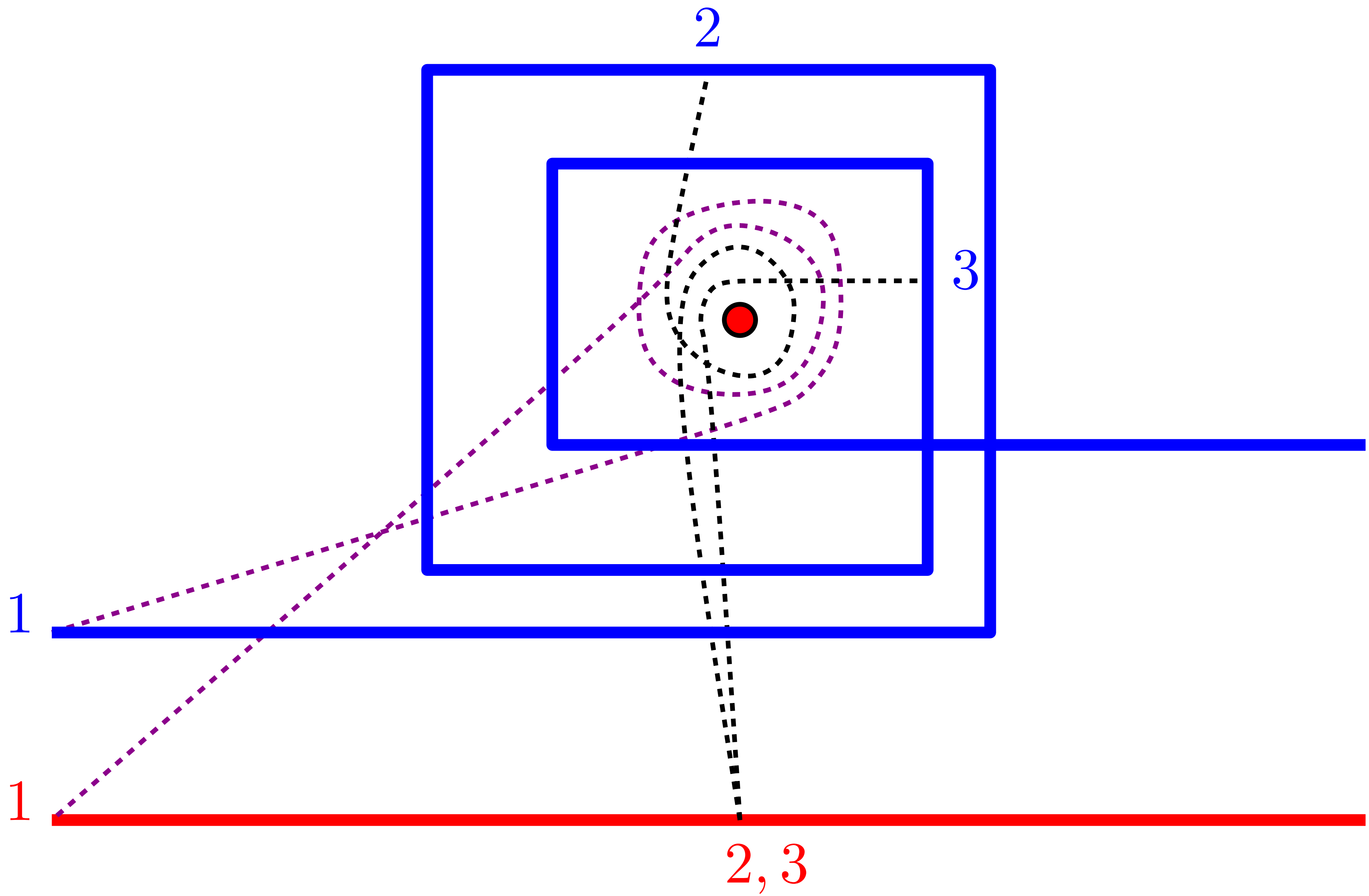

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