Walking Your Dog in the Woods in Polynomial Time

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Given two input curves, the *Fréchet distance*, sometimes called the *dog-leash distance*, between them is defined as the minimum length of a leash required to connect a dog and its owner as they walk without backtracking along their respective curves from one endpoint to the other. Fréchet distance is used as a measure of similarity of the two curves in many different applications [1, 2].

When the two curves are embedded in a general metric space, the distance between two points on the curves (the length of the shortest leash joining them) is not the Euclidean distance but a geodesic distance. For instance, this is the case if the space containing the two curves has obstacle regions which the leash cannot penetrate [3] or if the two curves lie on a terrain (or any surface) [4]. The definition of the classical Fréchet distance allows the leash to switch discontinuously, without penalty, from one side of an obstacle or a mountain to another.

We introduce a continuity requirement on the motion of the leash. We require that the leash cannot switch, discontinuously, from one geodesic to another; in particular, the leash cannot jump over obstacles and can sweep over a mountain only if it is long enough. We define the *homotopic Fréchet distance* between two curves as the Fréchet distance with this additional continuity requirement.

We expect that homotopic Fréchet distance is more useful in several if not all applications that use the classical Fréchet distance. In graphics, the continuous motion of the leash, which defines a correspondence between the two curves, can be used to morph between the two curves—two points joined by a leash morph into each other. In robotics, two motion sequences in the configuration space of a robot system may be more accurately compared by their homotopic Fréchet distance which accounts for obstacle regions.

Efficiently computing the homotopic Fréchet distance in general metric spaces is a new open problem. We present a polynomial-time algorithm for a special case of the general problem; namely, we compute the homotopic Fréchet distance between two polygonal curves in the plane with point obstacles.

Let S be a fixed Hausdorff metric space. A ***curve*** in S is (the image of) a continuous function from the unit interval $[0, 1]$ to S. A ***reparameterization of*** $\mathbf{[0, 1]}$ is a continuous, non-decreasing, onto function $\alpha\colon [0, 1] \to [0, 1]$. A reparameterization of a curve $A\colon [0, 1] \to S$ is any curve $A \circ \alpha$, where α is a reparameterization of $[0, 1]$. The ***length*** of any curve A, denoted $\mathrm{len}(A)$, is defined by the metric of S; in particular, two reparameterizations of the same curve have the same length.

A ***leash*** between two curves A and B is another curve $\lambda\colon [0, 1] \to S$ such that $\lambda(0) = A(s)$ and $\lambda(1) = B(t)$ for some parameters s and t. A ***homotopy*** between curves A and B is a continuous map $h\colon [0, 1] \times [0, 1] \to S$ such that $h(\cdot, 0) = A$ and $h(\cdot, 1) = B$. For any $t \in [0, 1]$, the one-parameter function $h(t, \cdot)$ is a leash from A to B. A ***leash map*** between curves A and B is a homotopy between some reparameterization of A and some reparameterization of B. Intuitively, a leash map describes the *continuous* motion of a leash between a dog walking along A and its owner walking along B. The ***length*** of a leash map ℓ, denoted $\mathrm{len}(\ell)$, is the maximum length of any leash $\ell(t, \cdot)$. Finally, the ***homotopic Fréchet distance*** between two curves A and B, denoted $\overline{\mathcal{F}}(A, B)$, is the infimum, over all leash maps ℓ between A and B, of the length of ℓ.

In spaces where shortest paths vary continuously as their endpoints move, such as the Euclidean plane, Fréchet distance and homotopic Fréchet distance are identical. In general, however, homotopic Fréchet distance could be larger (but never smaller) than classical Fréchet distance.

A ***homotopy relative to A and B***, or simply ***relative homotopy***, is a continuous function $h\colon [0, 1] \times [0, 1] \to S$, such that $h(\cdot, 0)$ and $h(\cdot, 1)$ are respectively of the form $A(u(\cdot))$ and $B(v(\cdot))$, where u and v are continuous functions from $[0, 1]$ to $[0, 1]$. Two leashes λ and λ' are ***relatively homotopic***, denoted $\lambda \simeq \lambda'$, if there is a relative homotopy h such

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that $h(0,\cdot) = \lambda$ and $h(1,\cdot) = \lambda'$. Any leash map is (the transpose of) a relative homotopy; thus, all leashes $\ell(t,\cdot)$ determined by a leash map ℓ lie in the same relative homotopy class.

We develop a polynomial-time algorithm to compute the homotopic Fréchet distance between two polygonal paths A and B in the ***punctured plane*** $\mathcal{E} = \mathbb{E}^2 \setminus P$, for some finite point set P. The points P act as obstacles; in any leash map in $\mathcal{E}$, the moving leash can neither touch nor jump over any obstacle point. See Figure 1 for examples.

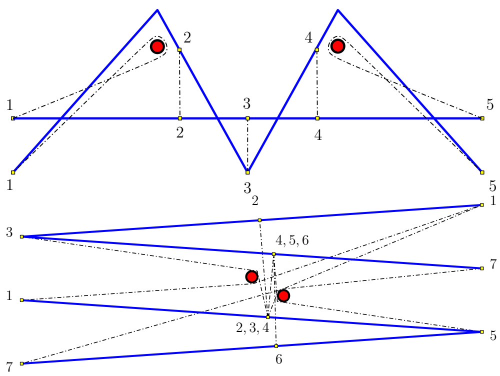

Figure 1: Two examples of optimum leash maps.

Let $\mathcal{F}_h(A,B)$ denote the *classical* Fréchet distance where distances are defined by shortest paths in $\mathcal{E}$ in the relative homotopy class h. We can show that the homotopic Fréchet distance is equal to the minimum, over all homotopy classes h, of $\mathcal{F}_h(A,B)$.

A ***proper line segment*** is a geodesic in $\mathbb{E}^2$ from a point in A to a point in B that does not touch any obstacle point in P. We say that a relative homotopy class is ***minimal*** if it contains a proper line segment. The key element in our algorithm is the following proposition.

Proposition. *Some optimal relative homotopy class contains a proper line segment from A to B.*

We enumerate all minimal relative homotopy classes as follows. Let $|A| = m$ and $|B| = n$. For every pair of points $p, q \in P$, we find all intersections of the line $\overline{pq}$ with A and B, in $O(m+n)$ time by brute force. For each pair of intersection points $a \in A$ and $b \in B$, we obtain four proper line segments arbitrarily close to ab. Altogether, we find $O(mn|P|^2)$ proper line segments, at least one in each minimal homotopy class, in $O(mn|P|^2)$ time.

There are polygonal curves and point sets that admit $\Omega(mn|P|^2)$ distinct minimal relative homotopy classes; see Figure 2 for an example. Thus, any improvement in this portion of the algorithm will require a finer characterization of optimal relative homotopy classes.

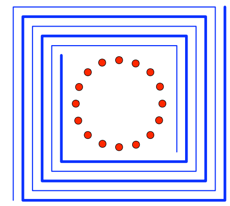

Figure 2: Curves and points with $\Omega(mn|P|^2)$ minimal relative homotopy classes.

We find an algorithm to compute the Fréchet distance $\mathcal{F}_h(A,B)$ in some relative homotopy class h. Our algorithm is based on Alt and Godau's algorithm for computing the classical Fréchet distance between polygonal paths in the plane [1], with several modifications. For example, shortest paths in our setting are not simple line segments, and so computing lengths of leashes is more complicated. Further details are omitted here due to lack of space.

Lemma. *Given a proper line segment in relative homotopy class h, the Fréchet distance $\mathcal{F}_h(A,B)$ can be computed in $O(mn|P|\log(mn|P|))$ time.*

Finally, to compute the homotopic Fréchet distance $\overline{\mathcal{F}}(A,B)$, we compute the Fréchet distance $\mathcal{F}_h(A,B)$ in each of the $O(mn|P|^2)$ minimal homotopy classes h. We conclude:

Theorem. *The homotopic Fréchet distance between two polygonal curves in the punctured plane can be computed in $O(m^2n^2|P|^3\log(mn|P|))$ time.*

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