TIGHT BOUNDS ON THE COMPLEXITY OF RECOGNIZING ODD-RANKED ELEMENTS

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ABSTRACT. Let $S = \langle s_1, s_2, s_3, ..., s_n \rangle$ be a given vector of n distinct real numbers. The *rank* of $z \in \mathbb{R}$ with respect to S is defined as the number of elements $s_i \in S$ such that $s_i \leq z$. We consider the following decision problem: determine whether the odd-numbered elements s_1, s_3, s_5, ... are precisely the elements of S whose rank with respect to S is odd. We prove a bound of $\Theta(n \log n)$ on the number of operations required to solve this problem in the algebraic computation tree model.

Let $S = \langle s_1, s_2, s_3, \ldots, s_n \rangle \in \mathbb{R}^n$ be a given vector. For an arbitrary real z, define the *rank* of z with respect to S, denoted by $\mathrm{rank}_S(z)$, as the number of elements of S less than or equal to z. Thus, for instance, the largest element of S has rank n. Let $\mathrm{odd}(S)$ denote the set of elements of S whose rank with respect to S is odd.

We consider the following problem: given a vector $S = \langle s_1, s_2, s_3, ..., s_n \rangle$ of n distinct real numbers, determine whether the odd-numbered elements $s_1, s_3, s_5, \ldots$ are precisely the elements of S whose rank with respect to S is odd. Without loss of generality, we can assume that n is even because, otherwise, we can append an extra element $+\infty$ without changing the answer.

We prove matching upper and lower bounds on the number of operations required to solve the problem in the algebraic computation tree model (see Ben-Or [2]).

The following algorithm solves the problem using $O(n \log n)$ comparisons. Sort $S' = \langle s_1, s_3, s_5, \ldots, s_{n-1} \rangle$ in non-decreasing order with an optimal sorting algorithm. Similarly, sort S in non-decreasing order. Then, scan the vector S' and the odd-numbered elements of S to decide whether the two are equal.

Next, we prove the matching lower bound.

For a vector $S = \langle s_1, s_2, s_3, \ldots, s_n \rangle$, let $\sigma(S)$ denote the permuted vector $\langle s_{\sigma(1)}, s_{\sigma(2)}, s_{\sigma(3)}, \ldots, s_{\sigma(n)} \rangle$. We call a permutation σ, where $\sigma(i)$ is odd if and only if i is odd, a *permissible* permutation.

Lemma 1. *There are* $\left(\left(\frac{n}{2}\right)!\right)^2$ *permissible permutations of a vector of* n *elements.*

Proof. There are $\frac{n}{2}!$ permutations of n elements that permute the $n/2$ odd-numbered elements only, and $\frac{n}{2}!$ that permute the $n/2$ even-numbered elements only. A permissible permutation of n elements is any composition of two permutations, one that permutes the odd-numbered elements only and one that permutes the even-numbered elements only. □

Observation 2. *A permutation* σ *is permissible if and only if its inverse* σ^{-1} *is permissible.*

Let $W \subset \mathbb{R}^n$ be the set of inputs for which the answer to the question posed in the problem is 'yes'. Recall that every point in W corresponds to a set of n distinct real numbers.

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Lemma 3. *For an arbitrary point $X \in W$, there is a unique permutation σ that sorts X, i.e., such that $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)} < \ldots < x_{\sigma(n)}$. Moreover, such a permutation σ is permissible.*

Proof. The uniqueness of the sorting permutation σ follows because every point in W corresponds to a set of distinct reals. When X is sorted, the odd-ranked elements must occupy the odd-numbered positions of the sorted vector. Since $X \in W$, the odd-ranked elements are already in odd-numbered positions of the original vector X. Therefore, the permutation σ is permissible. □

Let σ_X denote the sorting permutation for X.

Observation 4. *If σ_X is a permissible permutation, then $X \in W$.*

Lemma 5. *For every permissible permutation σ, there is a point $X \in W$ such that $\sigma = \sigma_X$.*

Proof. Let $X = \langle \sigma^{-1}(1), \sigma^{-1}(2), \sigma^{-1}(3), \ldots, \sigma^{-1}(n) \rangle$. We have,

$$\begin{aligned} \sigma(X) &= \langle \sigma(\sigma^{-1}(1)), \sigma(\sigma^{-1}(2)), \sigma(\sigma^{-1}(3)), \ldots, \sigma(\sigma^{-1}(n)) \rangle \\ &= \langle 1, 2, 3, \ldots, n \rangle \end{aligned}$$

Therefore, $\sigma(X)$ is sorted, and by Lemma 3, it is the unique permutation that sorts X; hence, $\sigma = \sigma_X$.

It remains to show that the point X that we chose belongs to W. The set of real numbers represented by X is $\{1, 2, 3, \ldots, n\}$. Since σ is permissible, so is σ^{-1} by Observation 2; hence, $\sigma^{-1}(i)$ is odd if and only if i is odd. Therefore, the ith component of the vector X is odd if and only if i is odd, which means that $X \in W$. □

Lemma 6. *For every two points $X, Y \in W$ such that $\sigma_X \neq \sigma_Y$, the two points X and Y lie in different connected components of W.*

Proof. Since $X, Y \in W$, both σ_X and σ_Y are permissible permutations, by Lemma 3.

For every point $A = \langle a_1, a_2, a_3, \ldots, a_n \rangle \in W$ such that

$$a_{\sigma_X(1)} < a_{\sigma_X(2)} < a_{\sigma_X(3)} < \ldots < a_{\sigma_X(n)}$$

we have $\sigma_A = \sigma_X$. Since σ_X is permissible, so is σ_A; by Observation 4, this implies that $A \in W$. Additionally, A is in the same connected component of W as X because every convex combination B of A and X satisfies $\sigma_B = \sigma_X$.

On the other hand, since $\sigma_Y \neq \sigma_X$, there exists an i in the range $1 \leq i \leq n-1$ such that $y_{\sigma_X(i)} \geq y_{\sigma_X(i+1)}$. Then, X and Y cannot be in the same connected component of W because they are separated by the hyperplane $y_{\sigma_X(i)} = y_{\sigma_X(i+1)}$; every point P on this hyperplane lies outside W because it corresponds to an input where P has fewer than n distinct elements.

We have thus shown that the region R_X where

$$R_X = \{\langle a_1, a_2, a_3, \ldots, a_n \rangle \in W : a_{\sigma_X(1)} < a_{\sigma_X(2)} < a_{\sigma_X(3)} < \ldots < a_{\sigma_X(n)}\}$$

is a maximal connected component of W containing X (R_X also happens to be convex); since $\sigma_Y \neq \sigma_X$, the region R_X does not contain Y. □

Theorem 7. *The set W has $\left(\left(\frac{n}{2}\right)!\right)^2$ connected components.*

Proof. The set W can be partitioned such that each part corresponds to a permissible permutation σ; by Lemma 5, $\sigma = \sigma_X$ for some $X \in W$. By Lemma 1, W is partitioned into $\left(\left(\frac{n}{2}\right)!\right)^2$ parts. By Lemma 6, every two distinct permissible permutations σ and σ' correspond to two different connected components of W, one consisting of all points

$X \in W$ for which $\sigma_X = \sigma$ and the other consisting of all points $Y \in W$ for which $\sigma_Y = \sigma'$. ☐

Corollary 8. *Every algebraic computation tree that decides the membership problem in* W *must have depth* $\Omega(n \log n)$.

Proof. Ben-Or [2] has proved that the minimum height of an algebraic computation tree deciding membership in W is $\Omega(\log {}^{\#}W)$ where ${}^{\#}W$ is the number of connected components of W. By Theorem 7, such a tree must have depth $\Omega(n \log n)$. ☐

Note. A general result of Arge, Knudsen, and Larsen [1] implies a lower bound for the current problem of $\Omega\left(\frac{n}{B} \log_{M/B} \frac{n}{B}\right) = \Omega(\textsc{Sort}(n))$ input-output operations in the external-memory comparison tree model with memory size M and block size B. The lower bound is tight since an I/O-efficient sorting algorithm can be used to solve the problem in $O(\textsc{Sort}(n))$ I/O's.

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