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Pants Decomposition of the Punctured Plane

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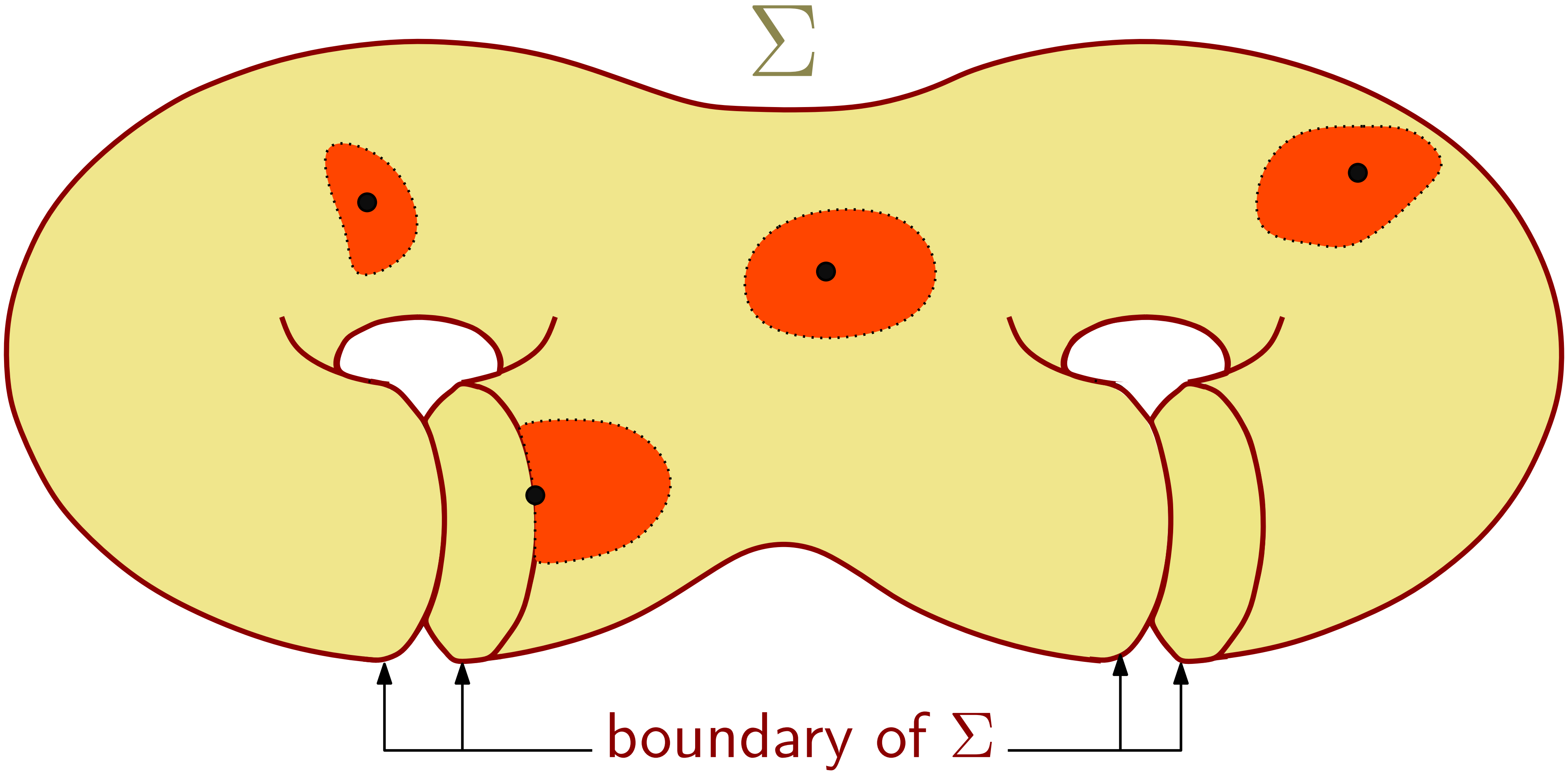
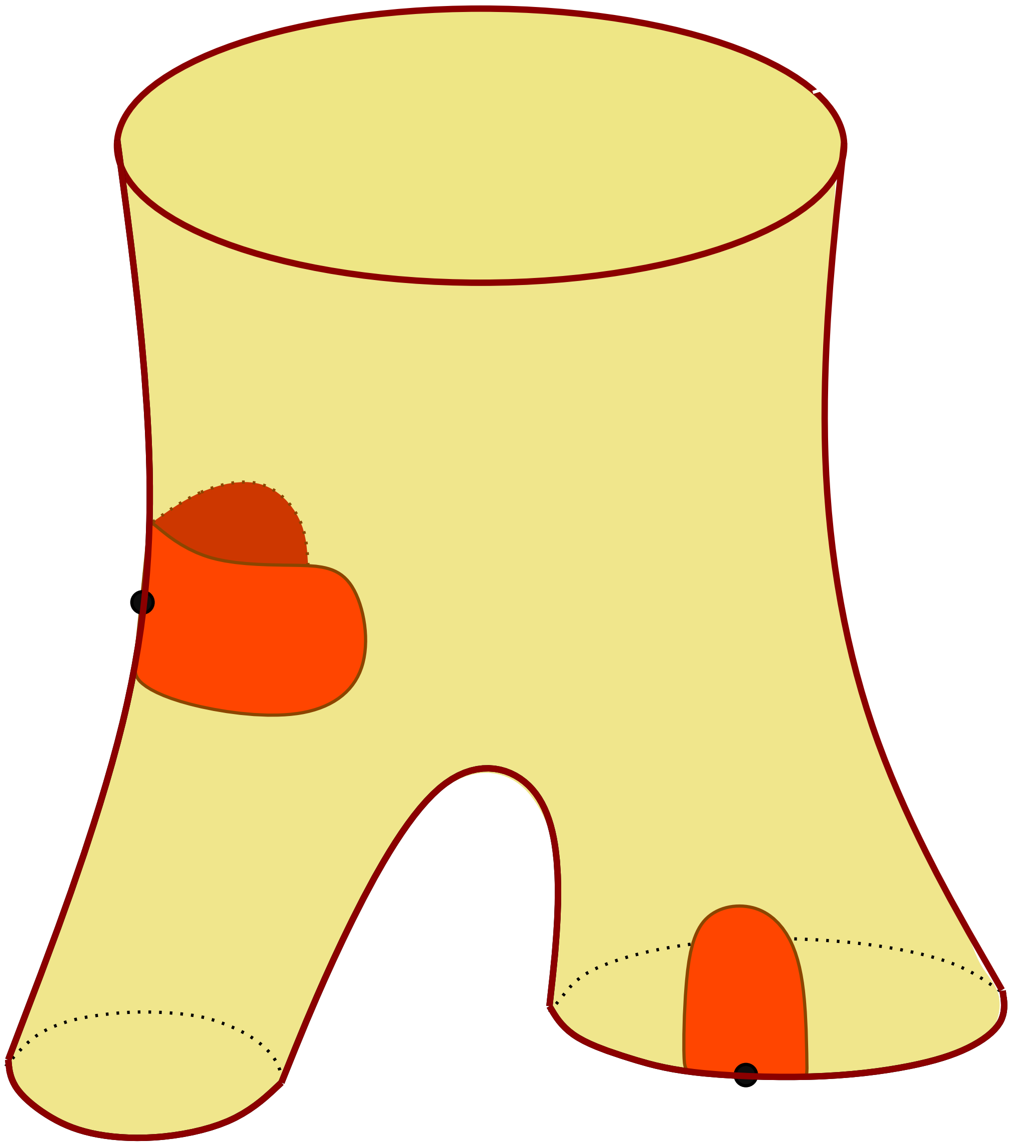
Joint work with Sheung-Hung Poon

Surface $\equiv$ 2-manifold



Pant $\equiv$ a sphere with 3 holes



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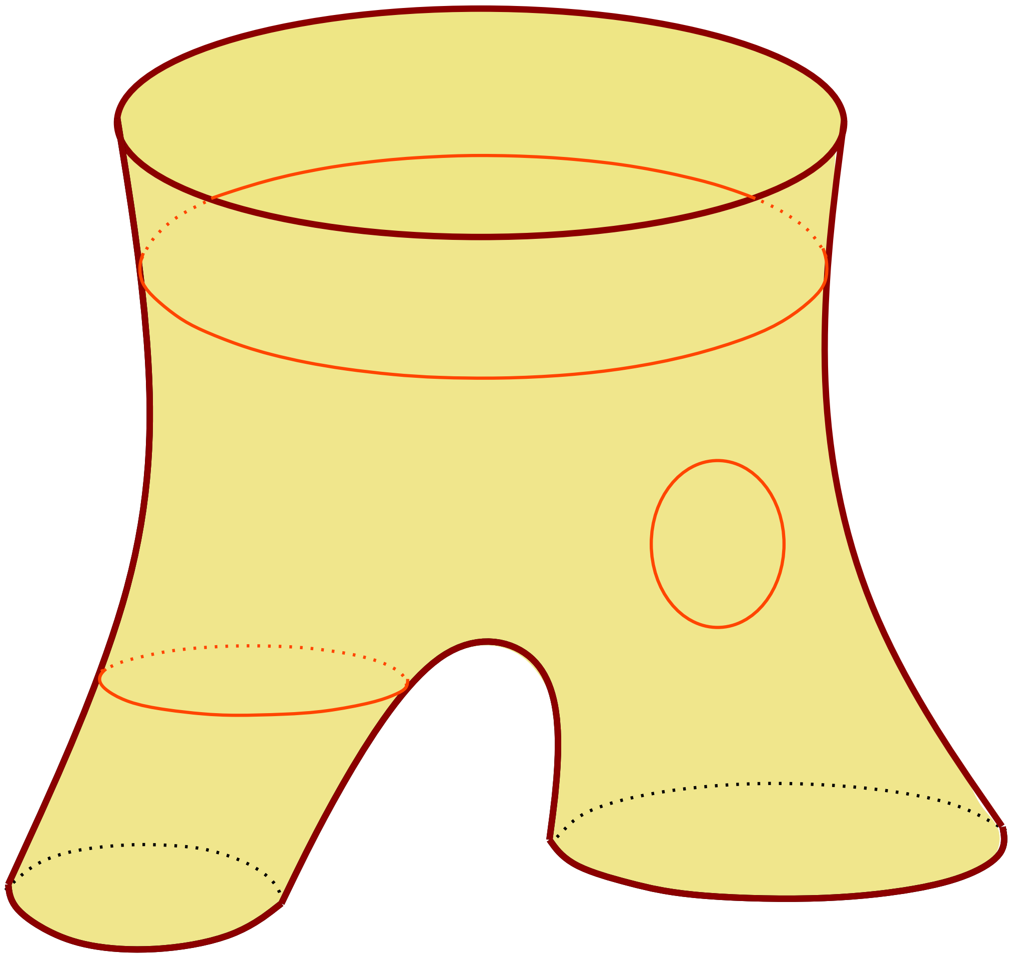


Every simple cycle on a pant is contractible to a point or to a boundary

Pants decomposition

Definition: A set of simple cycles that decompose a surface into disjoint pants

To understand the topology of the surface and to compute its various properties

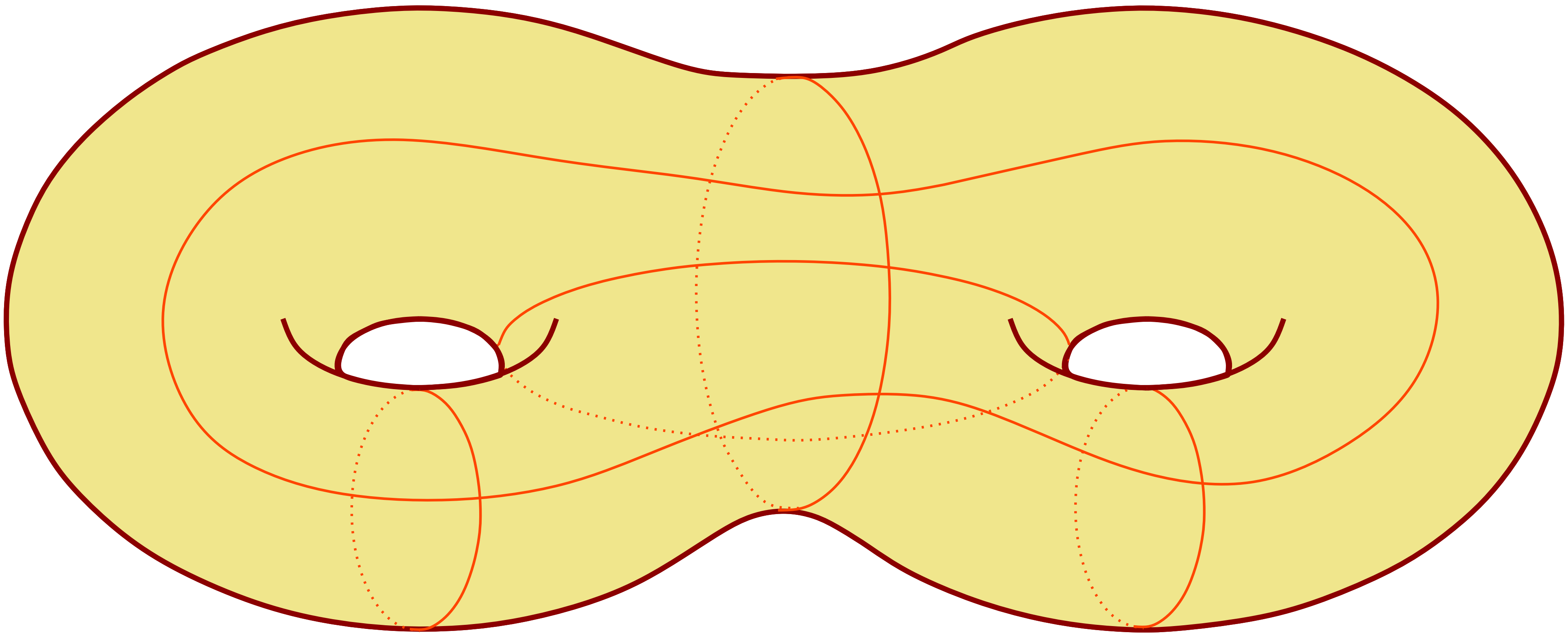
Every compact orientable surface has at least one pants decomposition*

** except sphere, cylinder, disk, or torus*

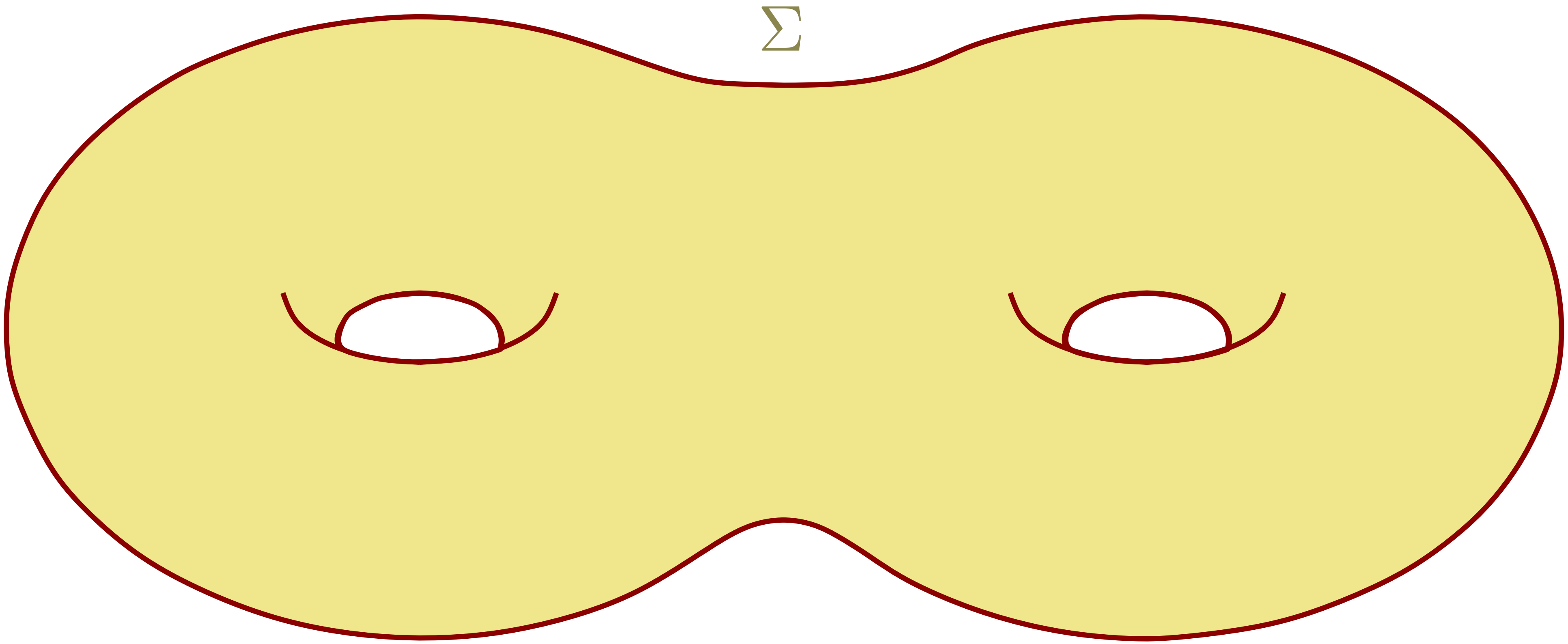
A surface can have many pants decompositions

Essential cycle

Simple cycle not contractible to a point or to a boundary



Example: decomposing into pants



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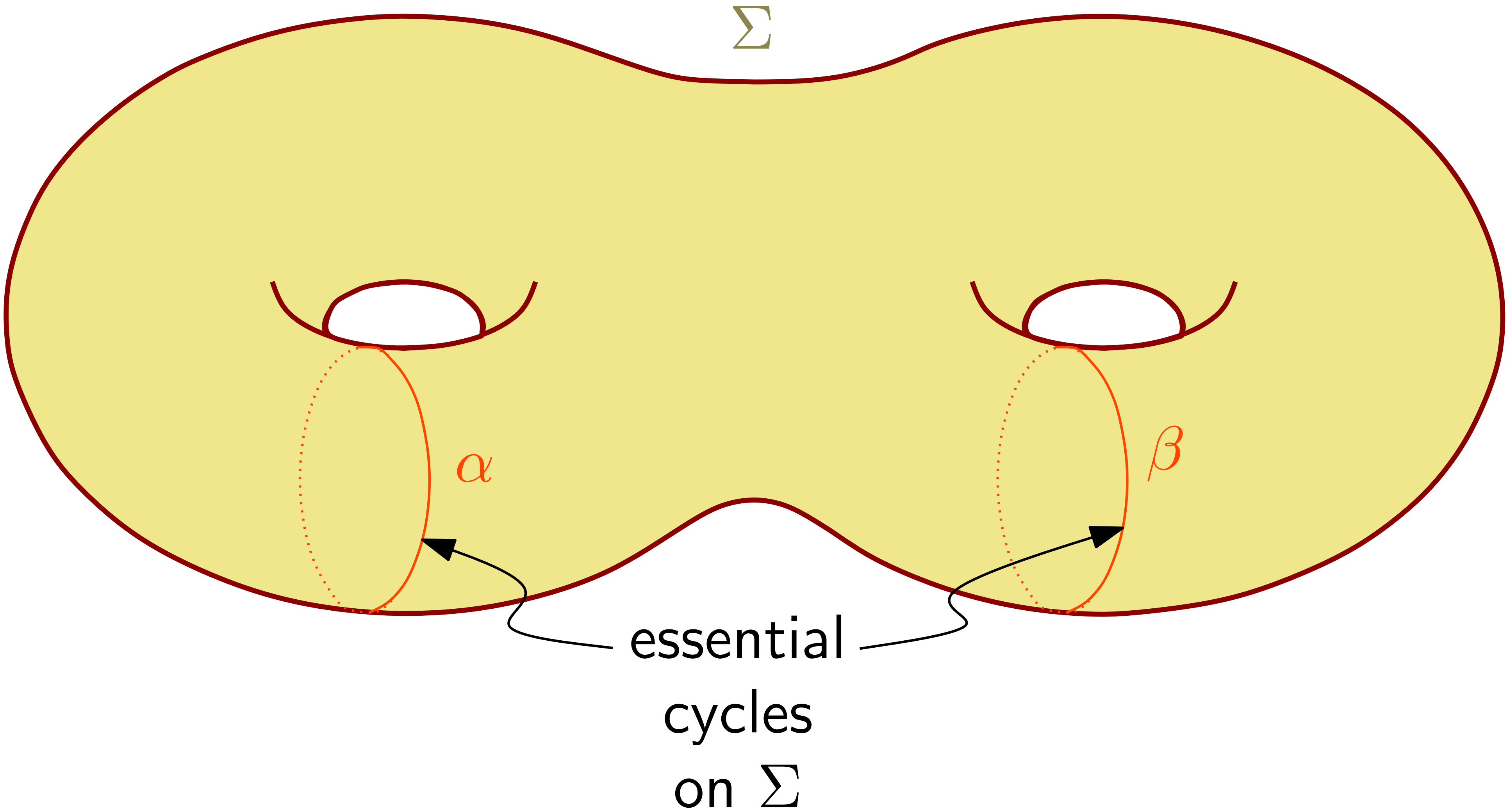

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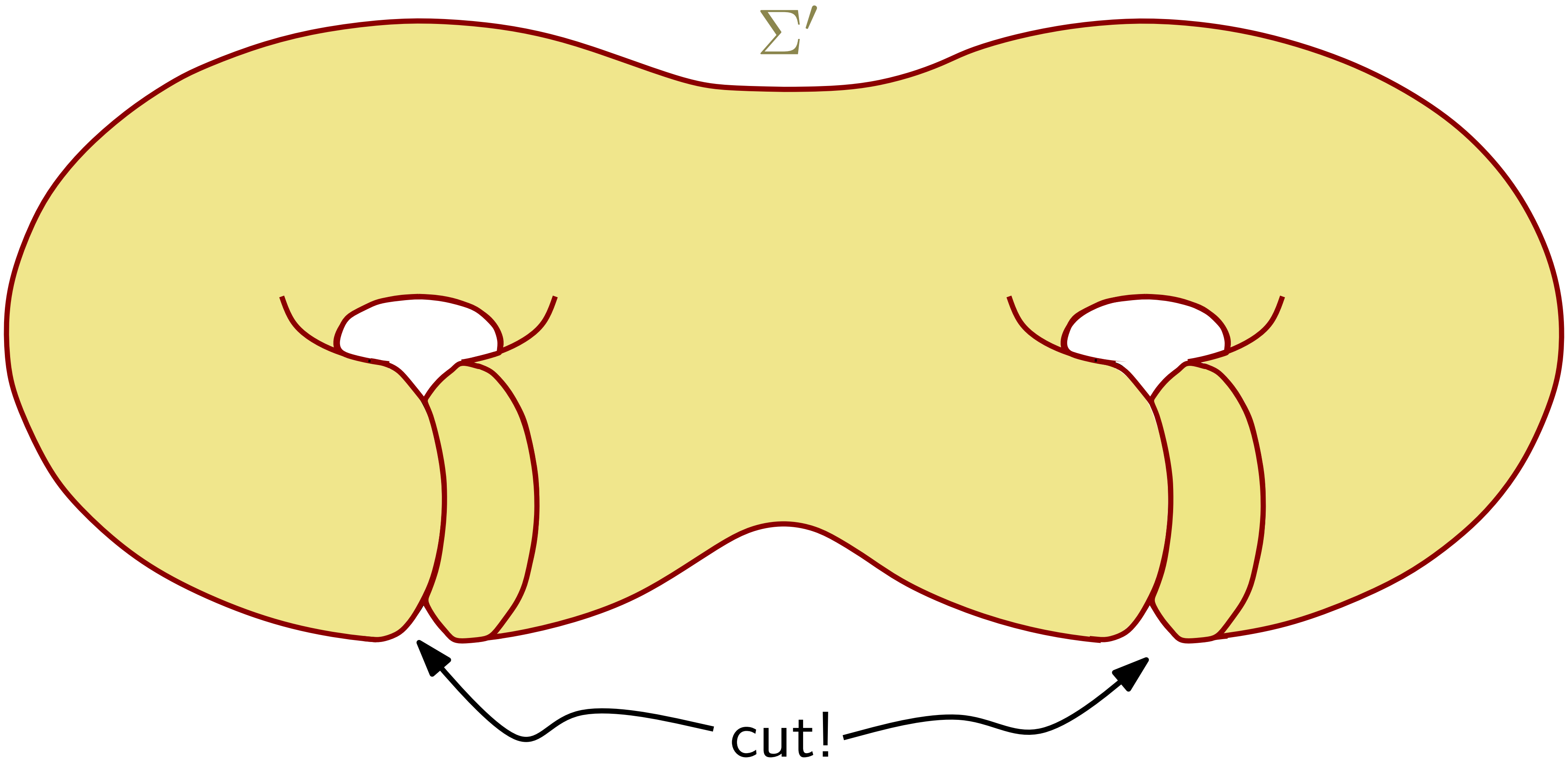

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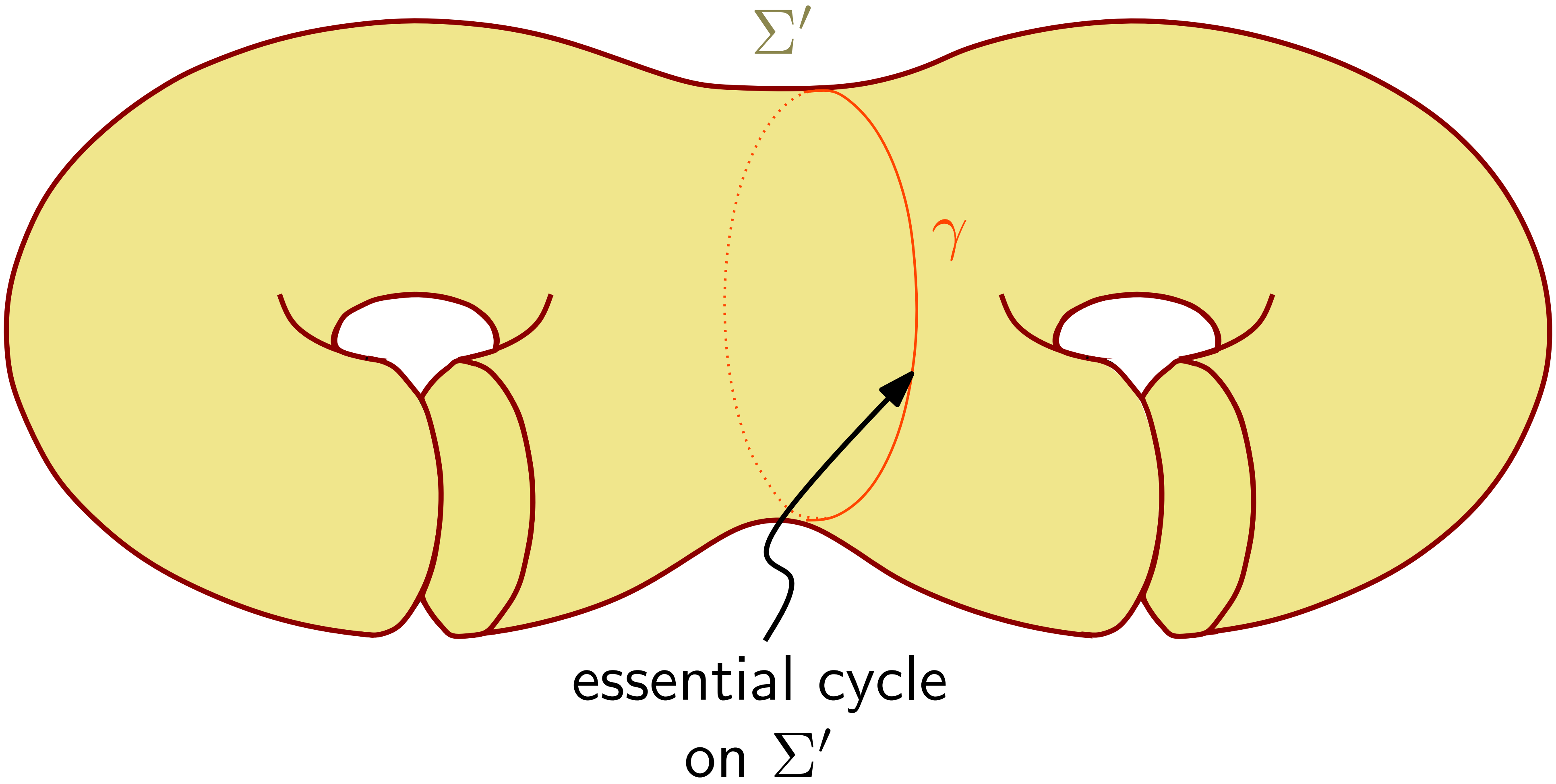

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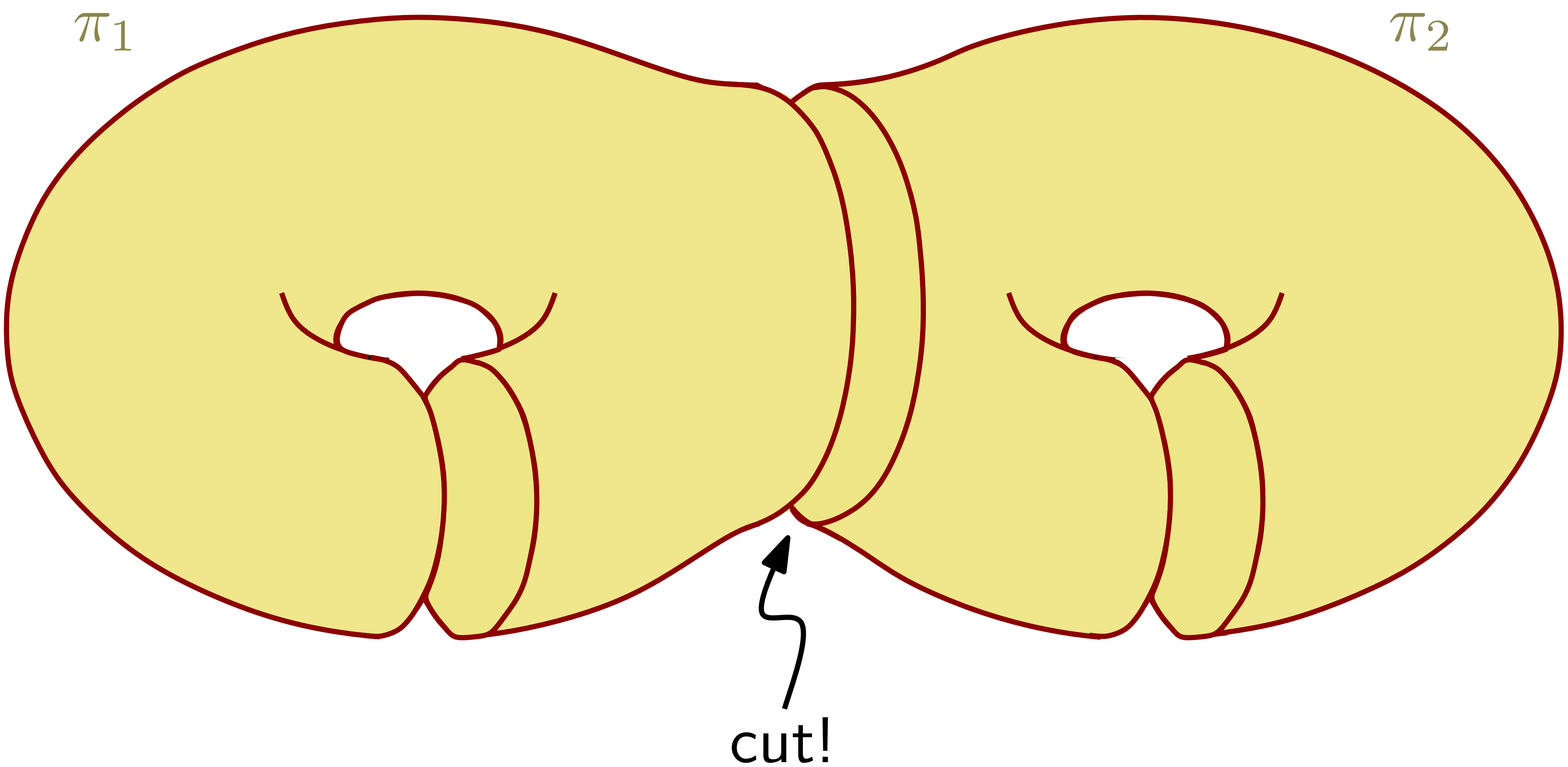

The big open problem

Computing an exact or approximate *shortest* pants decomposition of a general combinatorial surface

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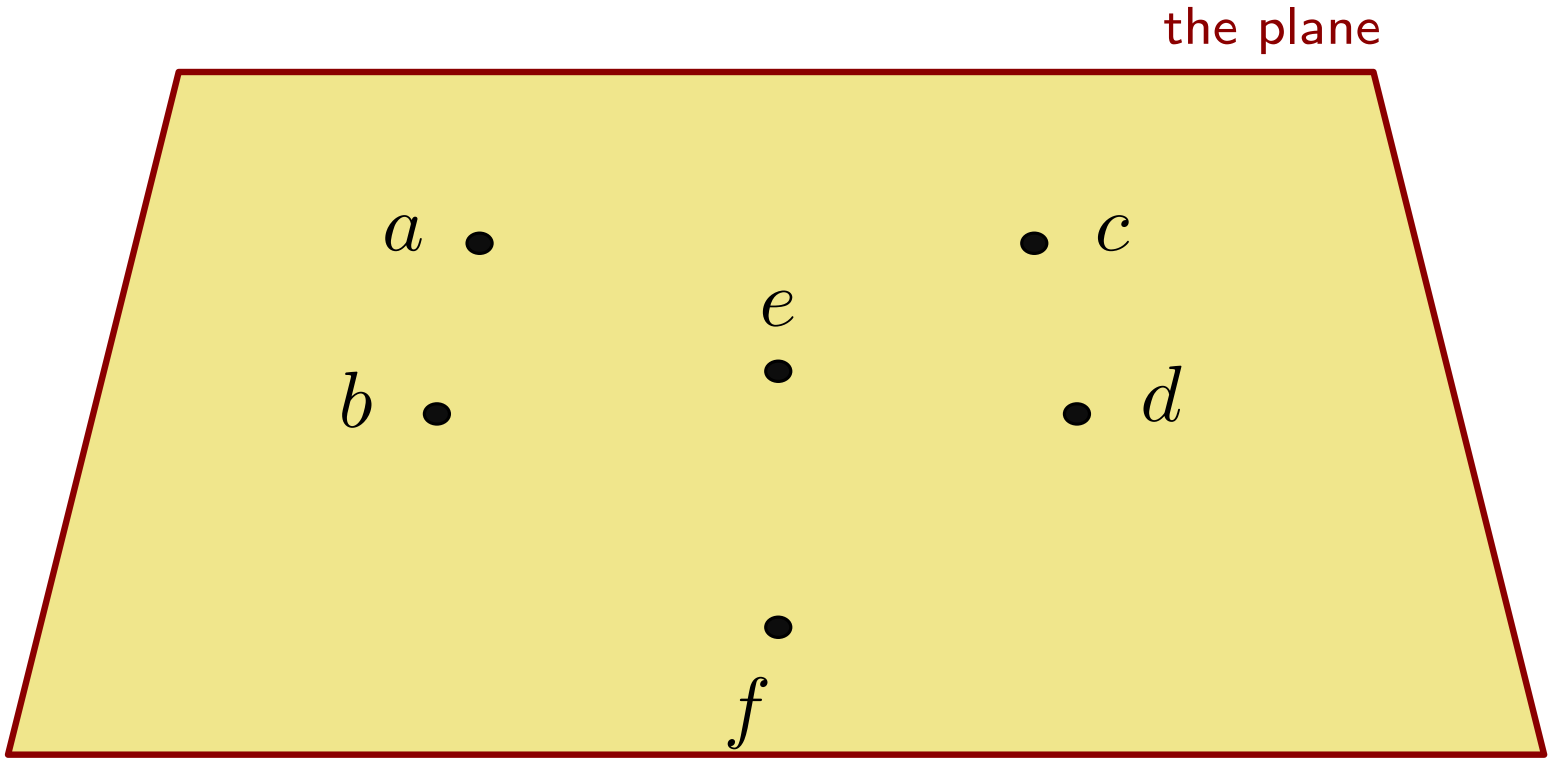
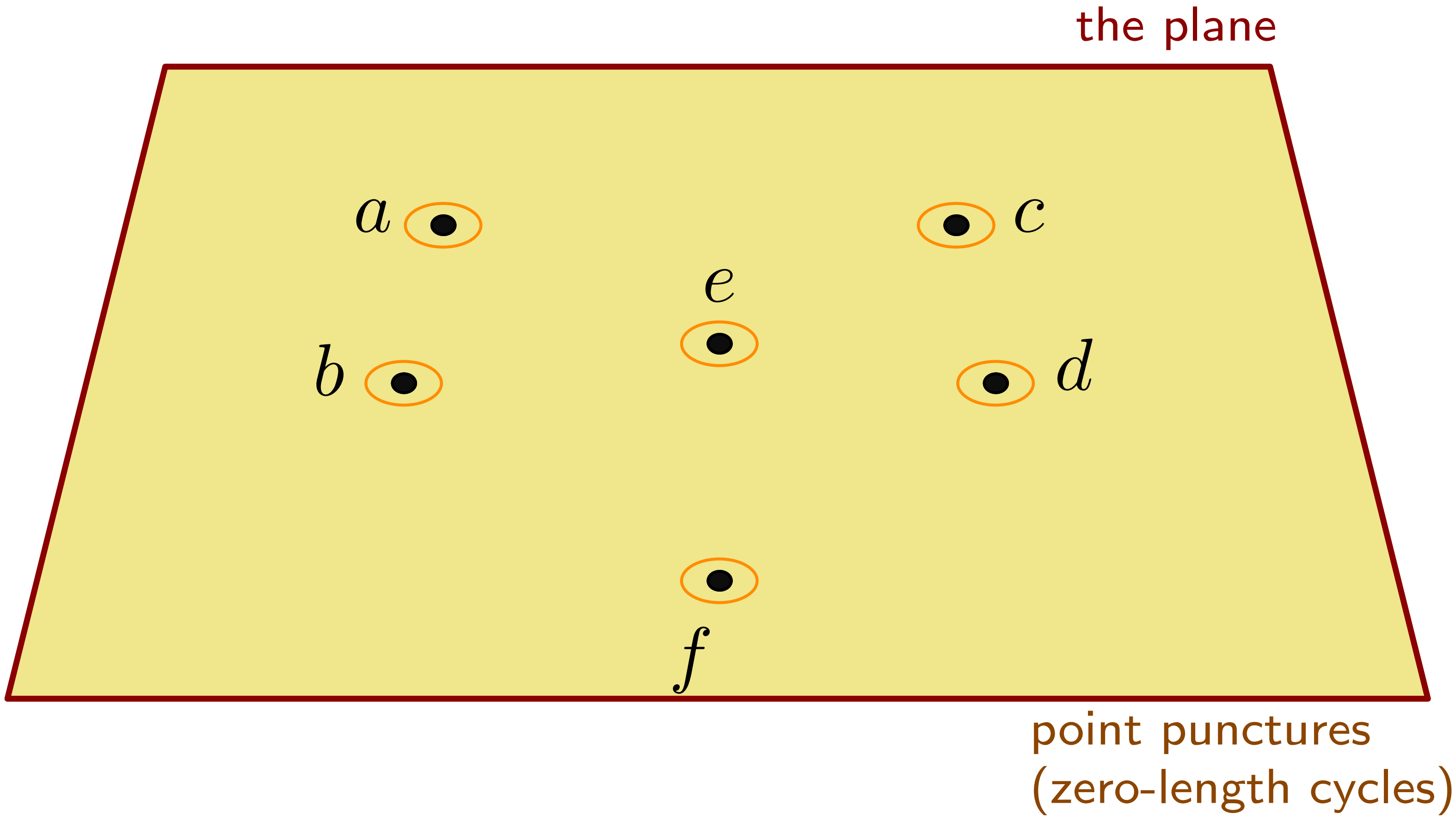
Computing an exact or approximate shortest pants decomposition of a general combinatorial surface

We consider a variant in the Euclidean plane . . .

Punctured plane

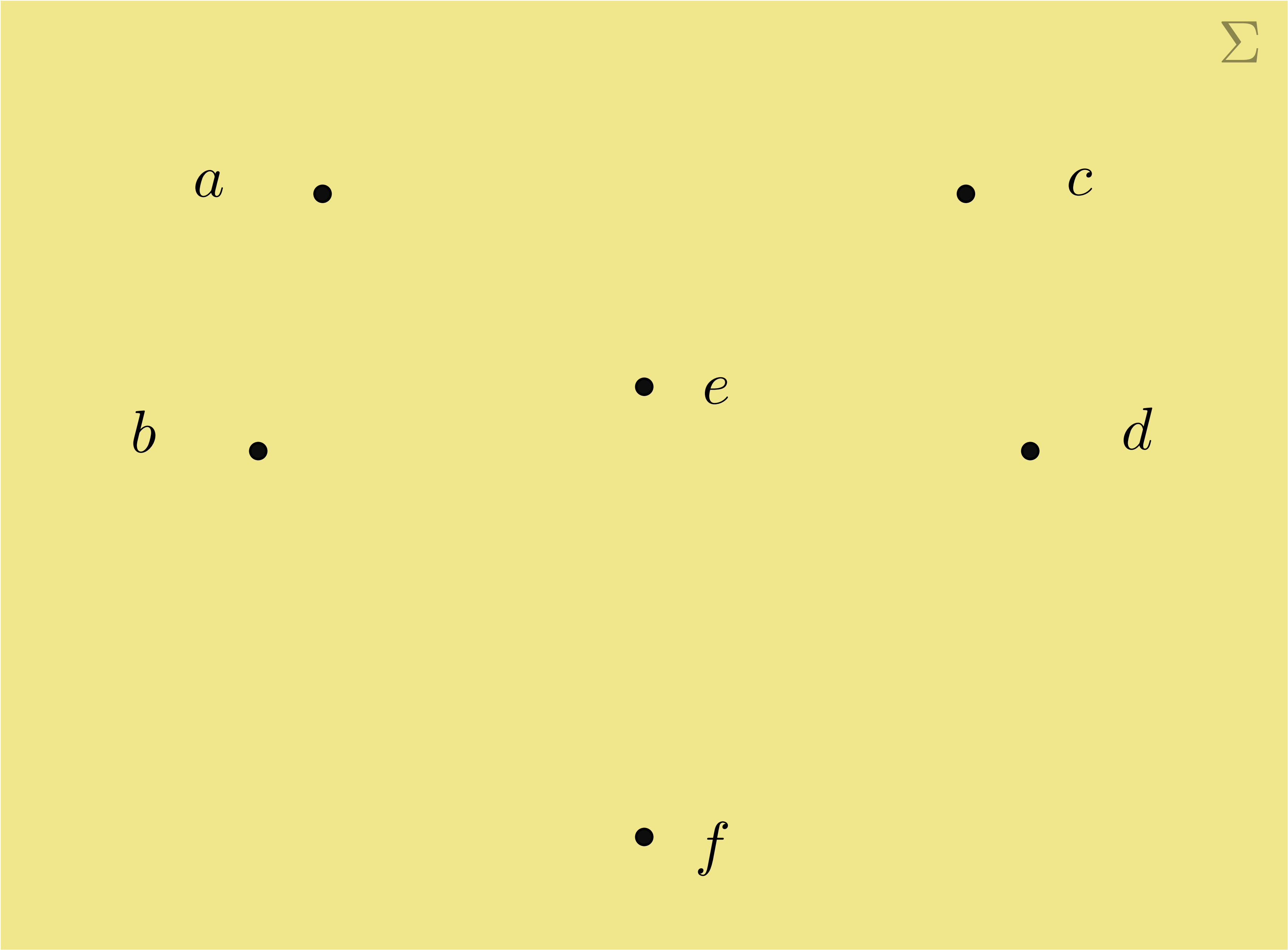


Punctured plane



Surface Σ is the plane minus n points

Decomposing the punctured plane



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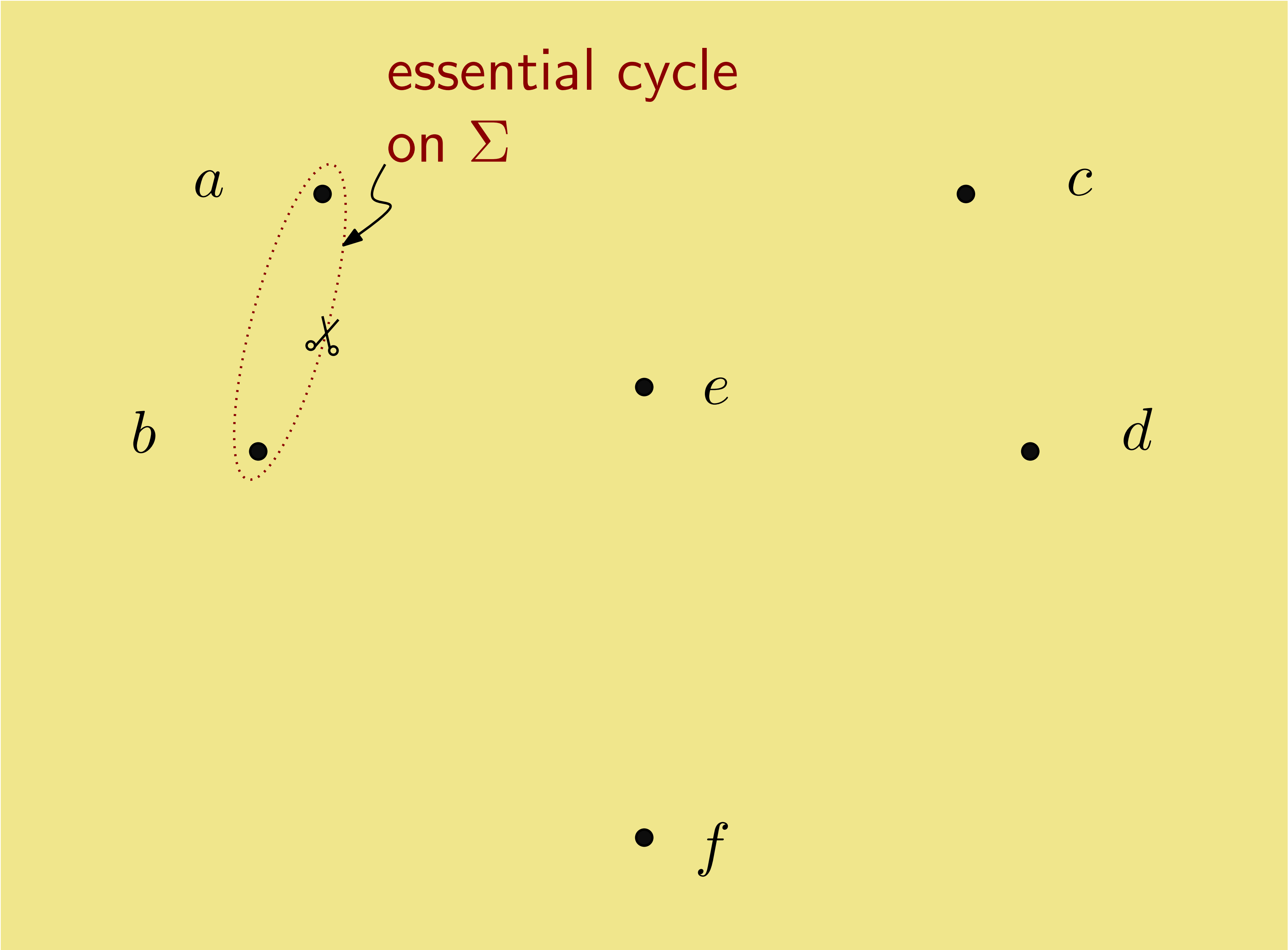
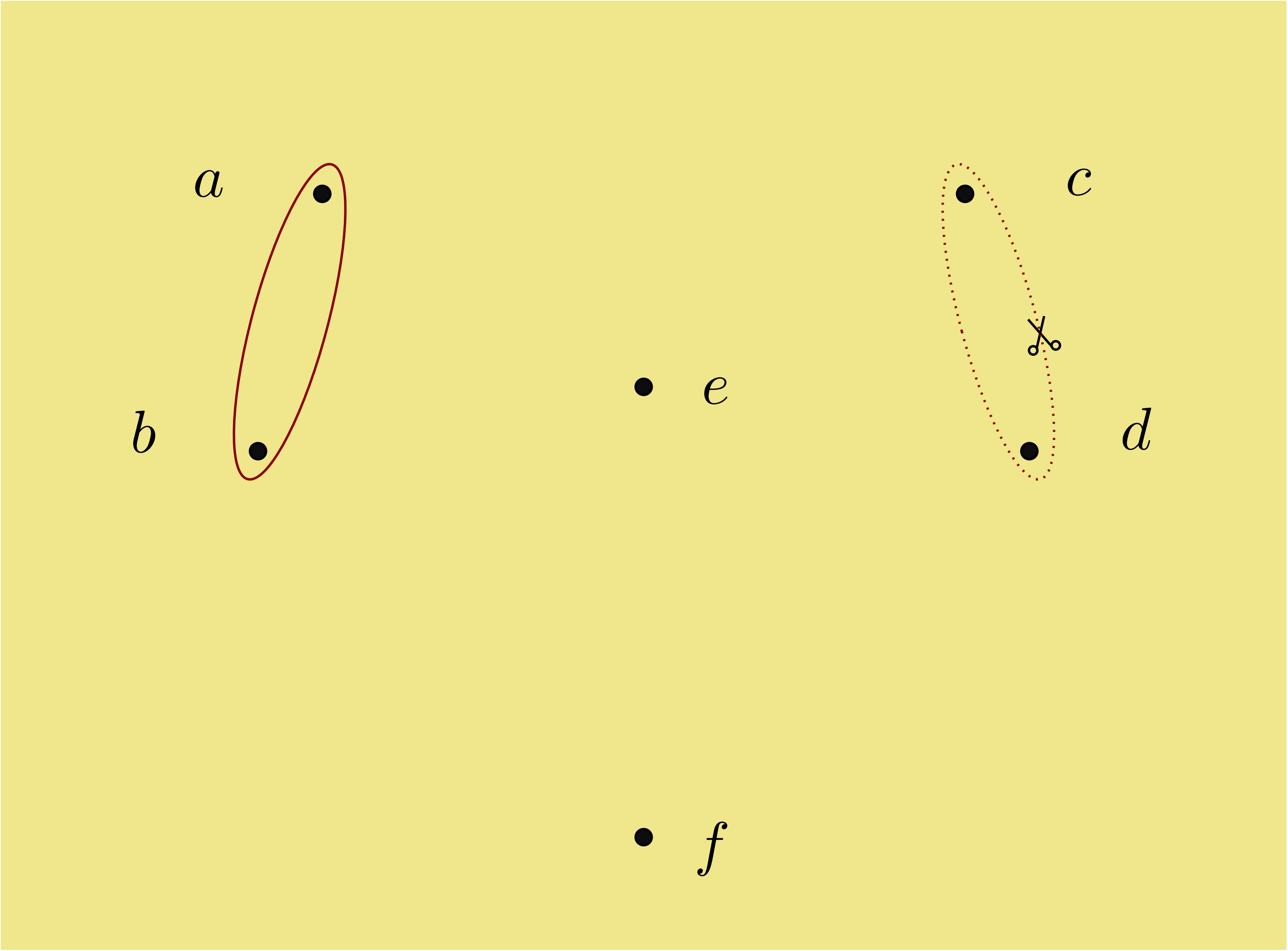


Decomposing the punctured plane



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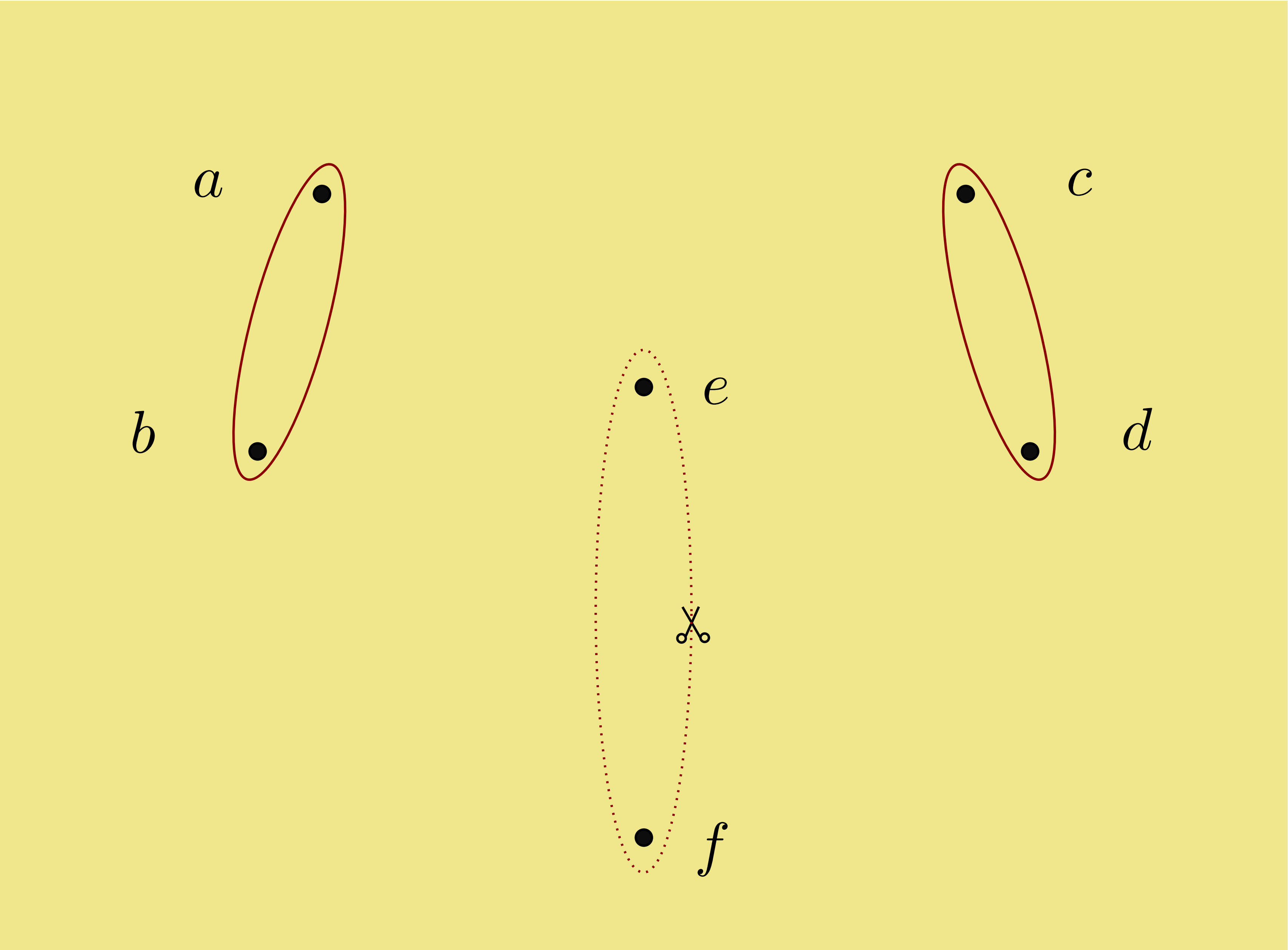

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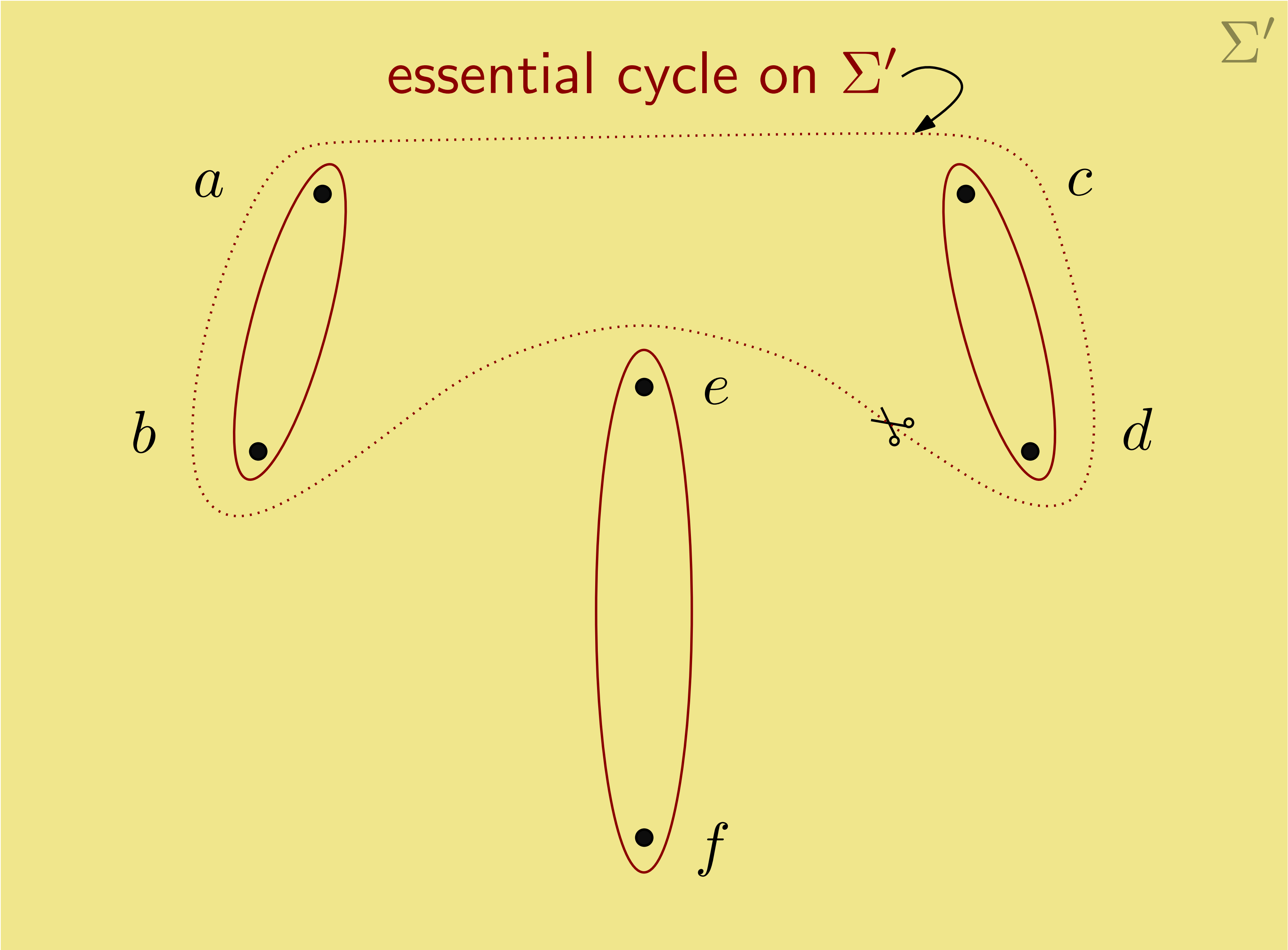

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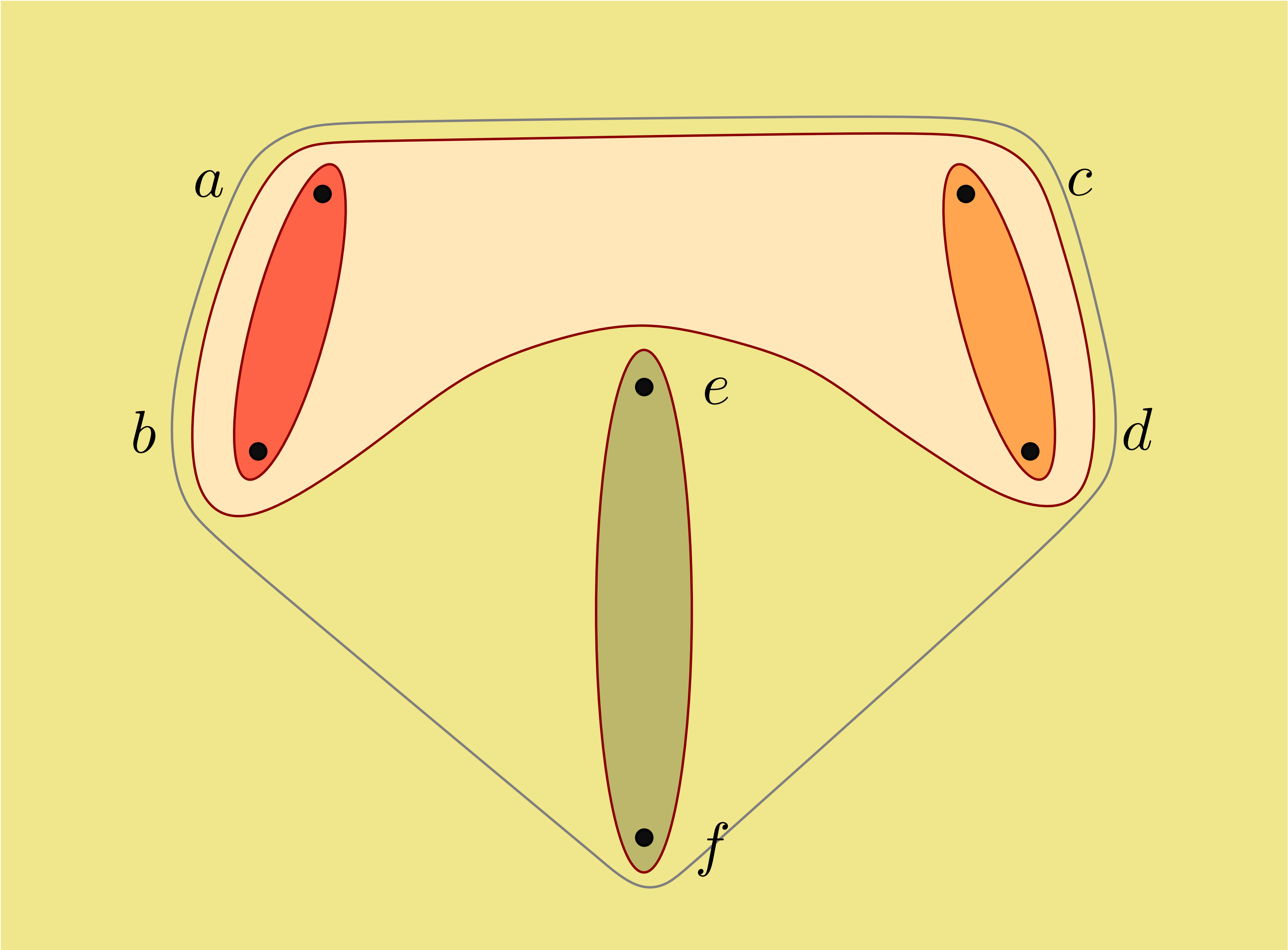
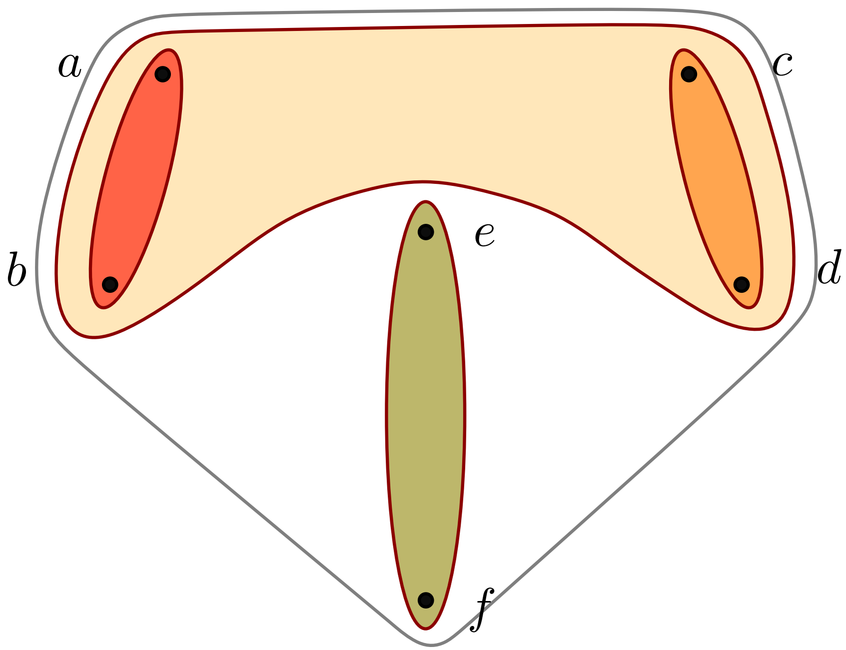
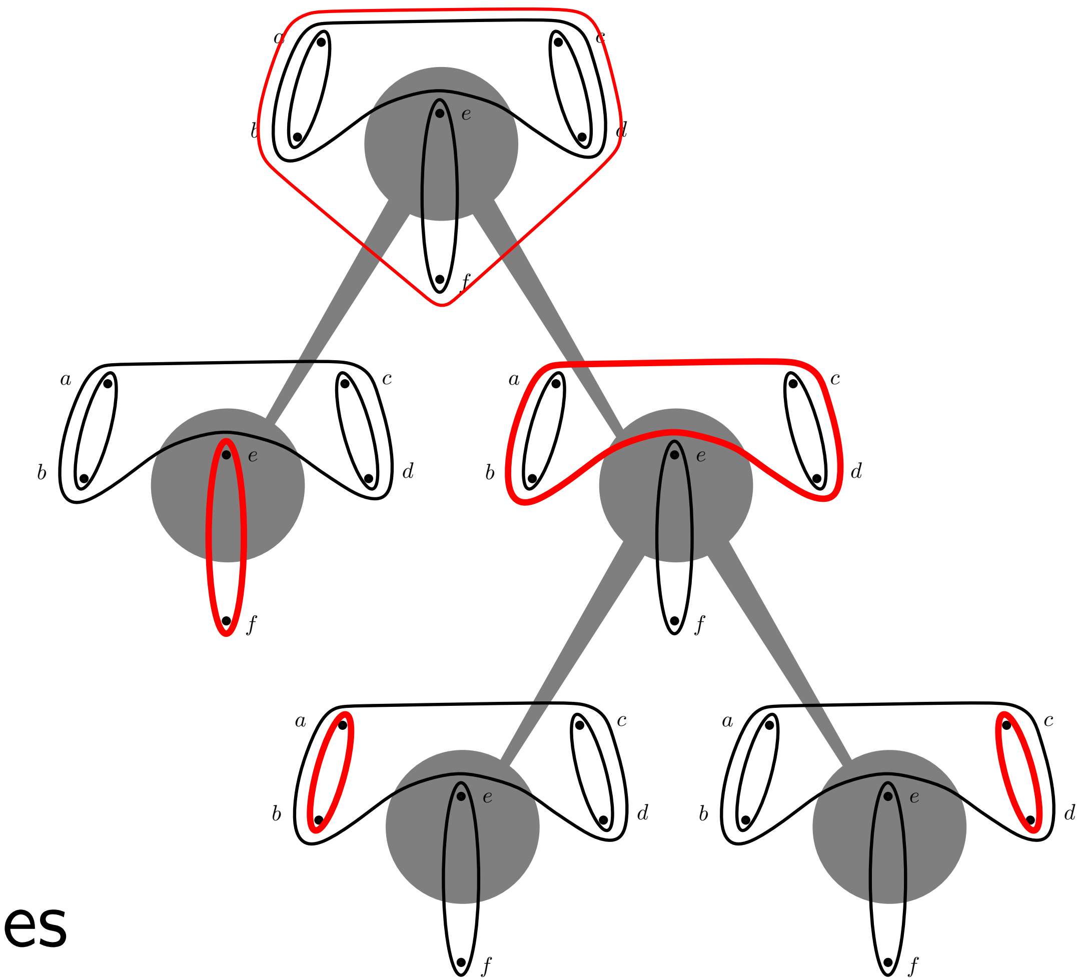


Properties





$n-1$ disjoint simple cycles

Nested in a binary tree

Decompose the plane into a set of pants and an unbounded component

Shortest pants decomposition

Input: A set P of n points in the plane $\mathbb{E}^2$

Remove an ε-disk D_i centered at each point $p_i \in P$

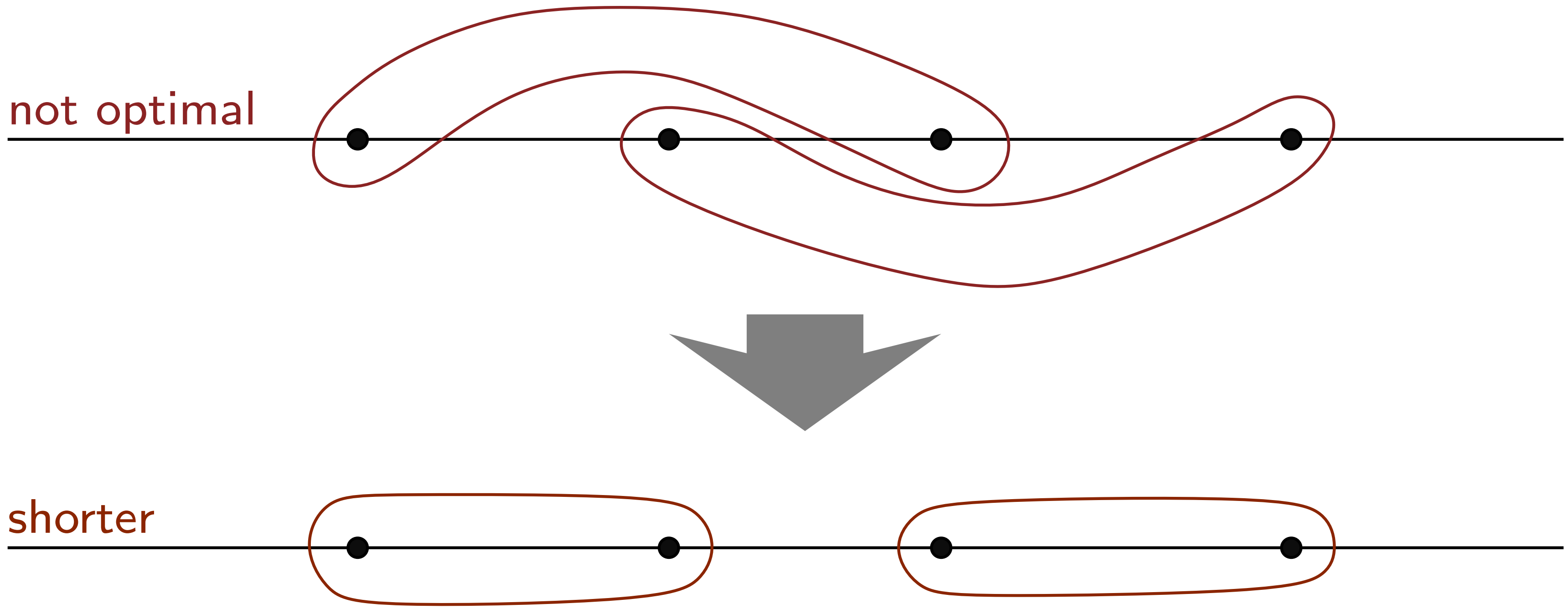
Let Π be a pants decomposition of $\mathbb{E}^2 \setminus \bigcup_{i=1}^{n} D_i$

Imagine tightening the cycles of Π as $\varepsilon \to 0$

The limit is a *non-crossing* pants decomposition Π'

Problem: Compute a non-crossing pants decomposition Π^* of $\mathbb{E}^2 \setminus P$ of minimum total length

Points on a line



Lemma: Every cycle in a shortest pants decomposition of collinear points encloses an *interval* of points

Compute shortest pants decomposition in $O(n^2)$ time using *dynamic programming* with Yao's speedup

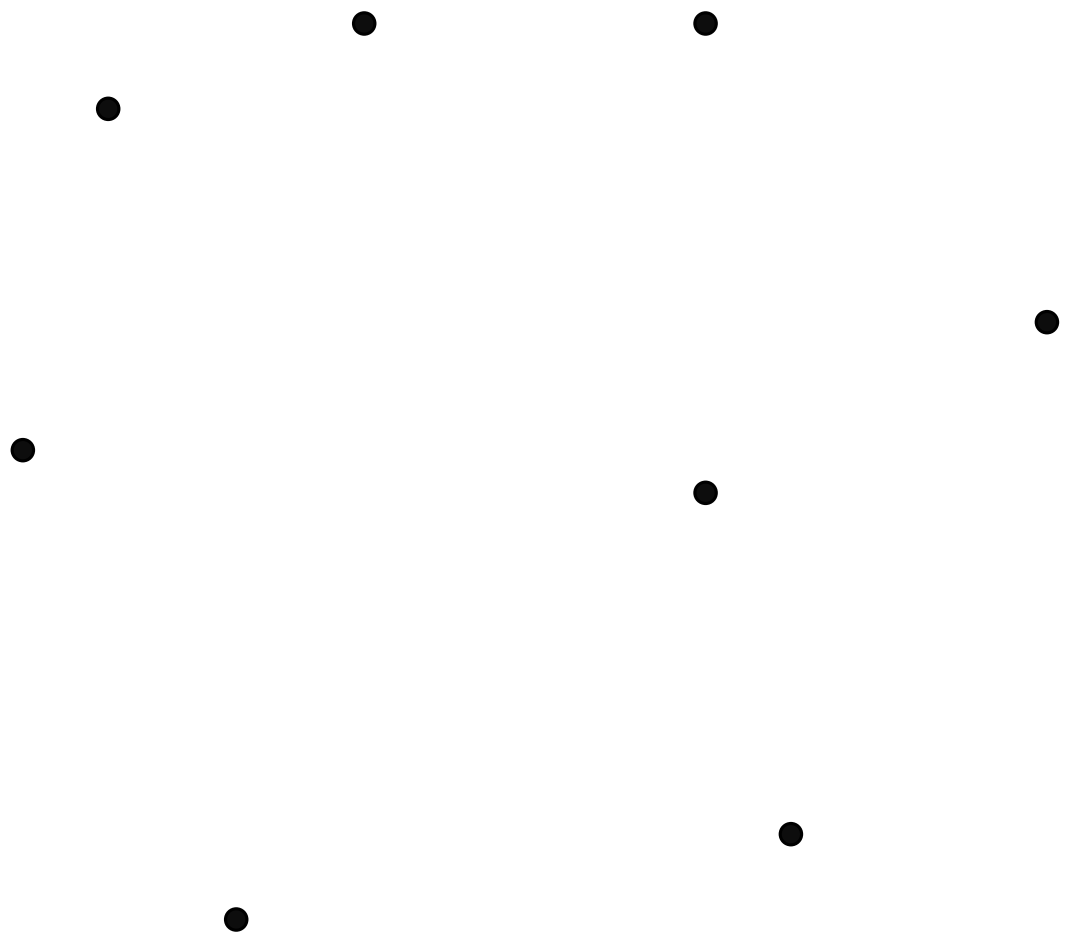
A lower bound

Every cycle in a shortest pants decomposition is a simple polygon with vertices in P
(no Steiner points)

A shortest pants decomposition Π^* of $\mathbb{E}^2 \setminus P$ contains a TSP tour of P

So, $|\Pi^*| \geq |TSP(P)|$

$O(\log n)$ approximation



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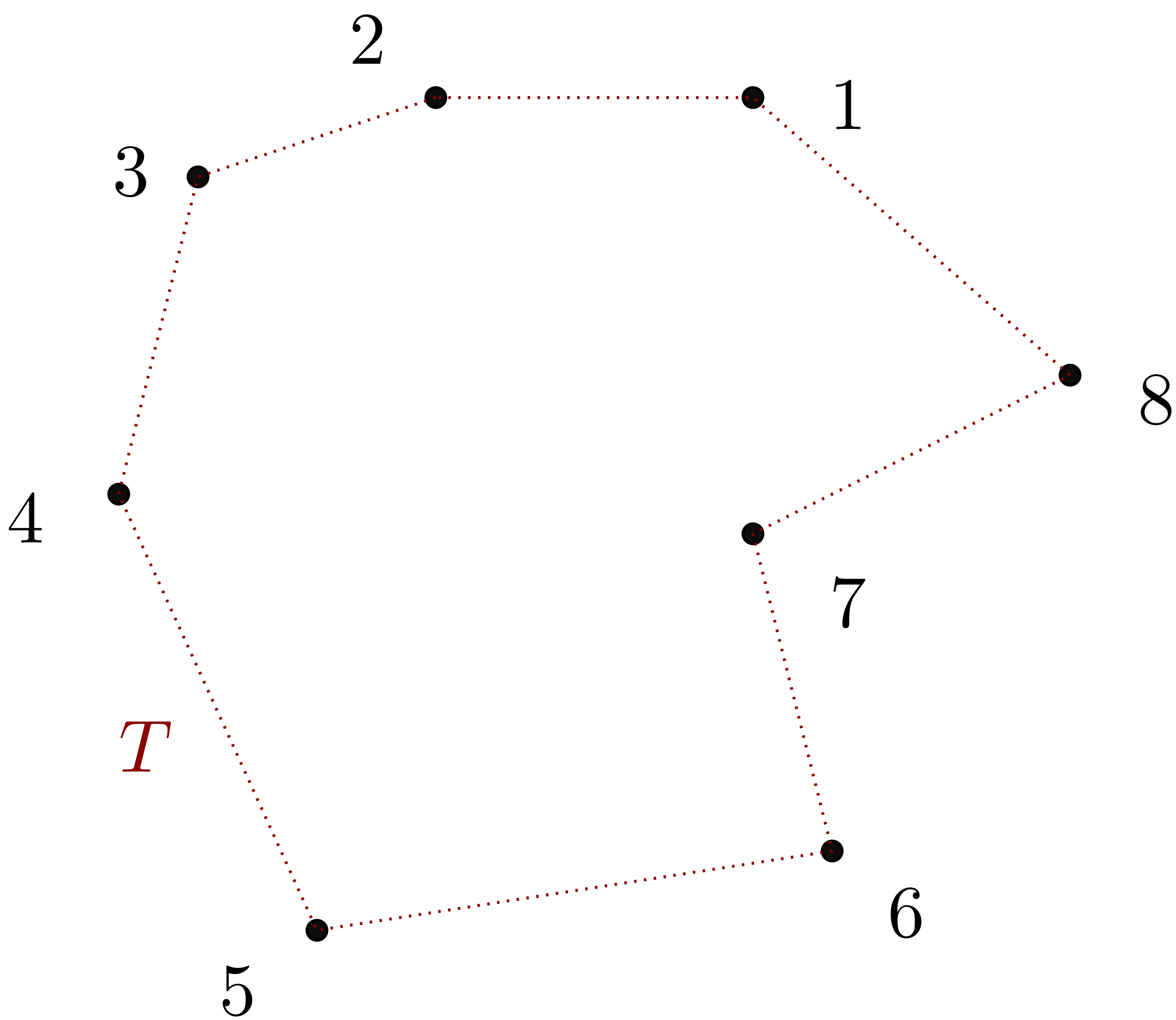
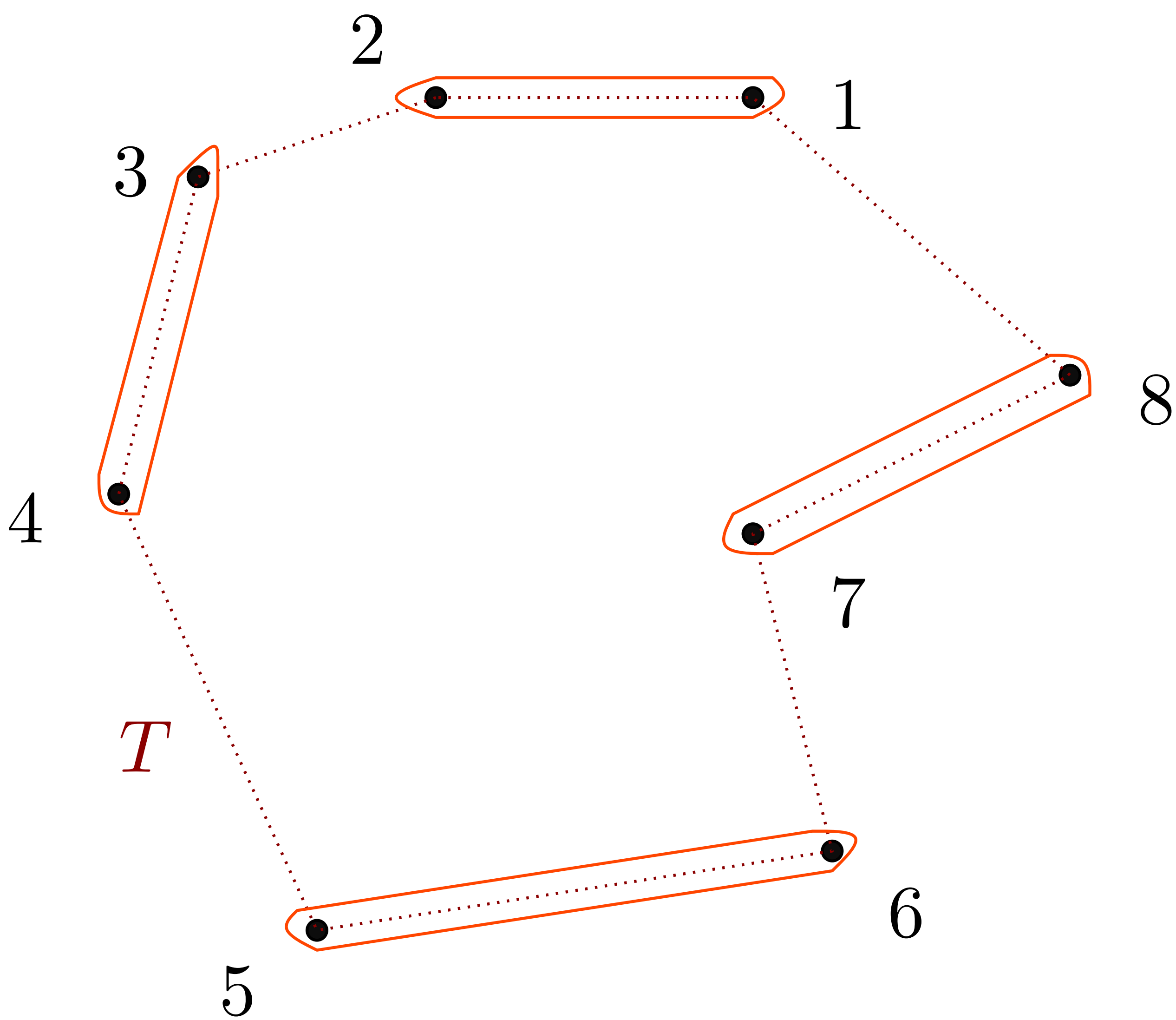


Construct an $O(1)$-approximate TSP tour T

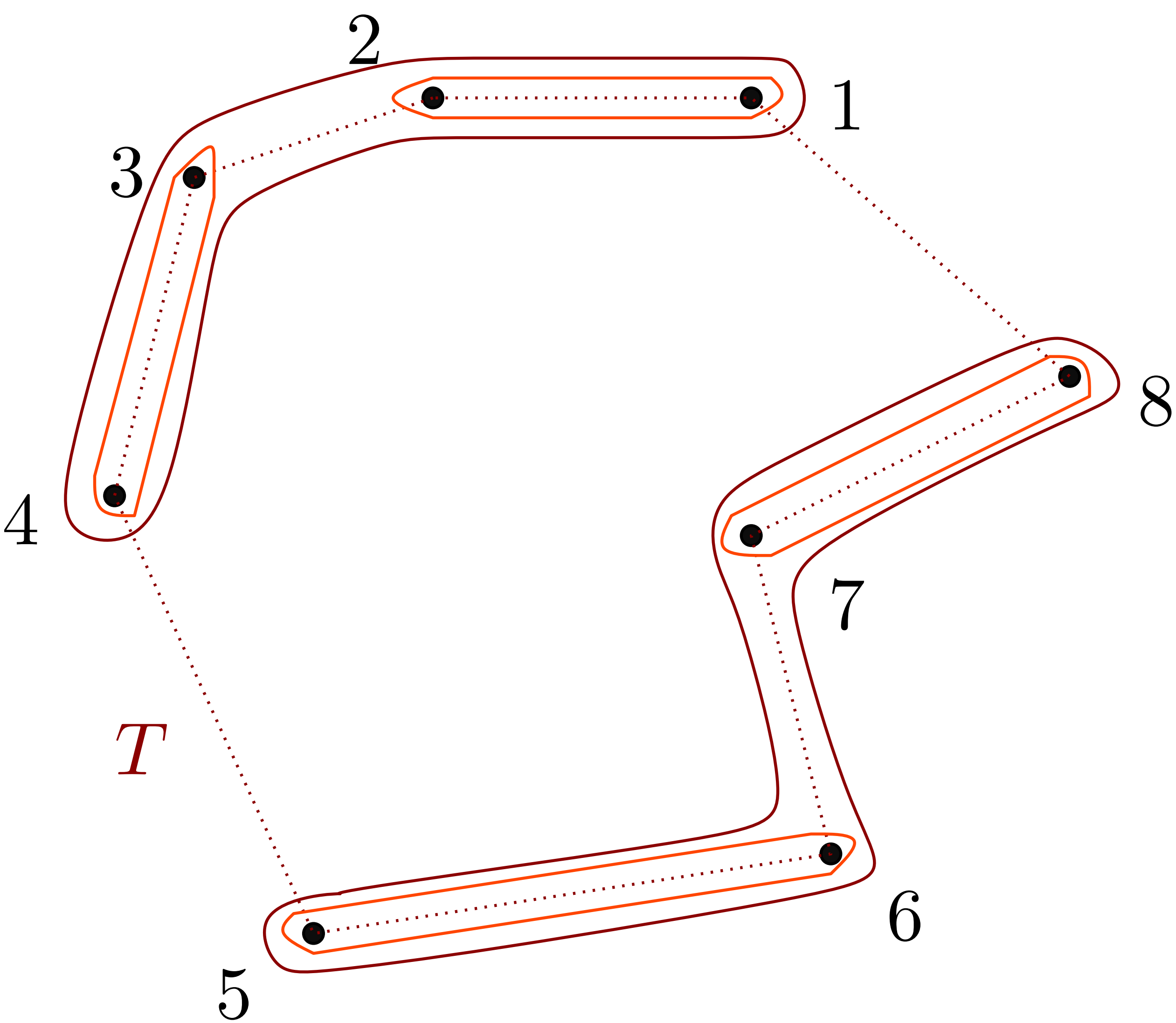
Start with the n points in order along the tour T

$O(\log n)$ approximation



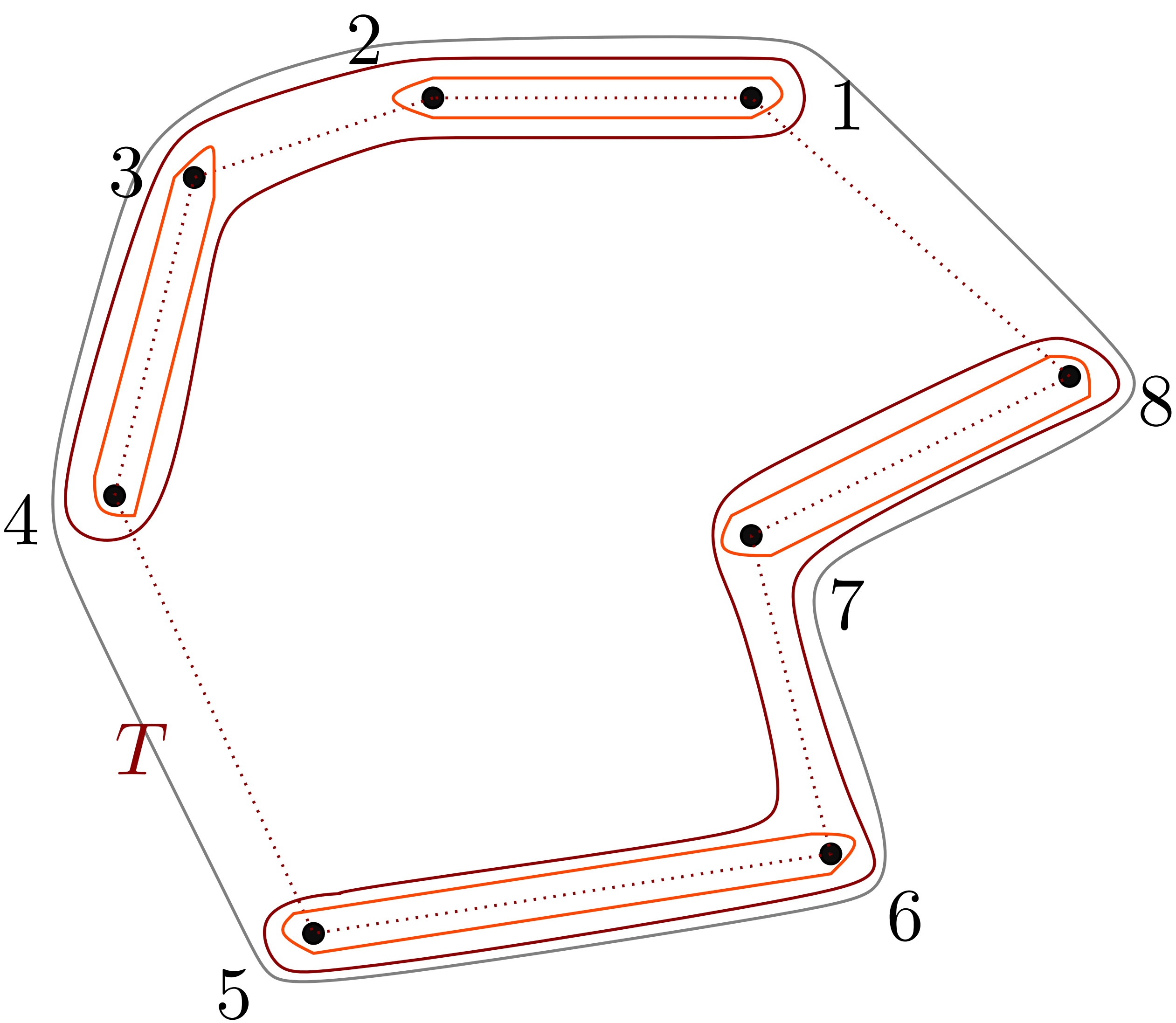
Repeatedly enclose pairs of smaller cycles by a bigger cycle until we have a pants decomposition Π

$O(\log n)$ approximation



Each cycle of Π is obtained by doubling the edges of a sub-tour of T

$O(\log n)$ approximation



Each edge of T belongs to $O(\log n)$ cycles of Π

So,

$$|\Pi| \leq O(\log n)\,|T| \\ \leq O(\log n)\,|\Pi^*|$$

PTAS

For every $\varepsilon > 0$, compute a $(1+\varepsilon)$-approximate shortest pants decomposition in polynomial time

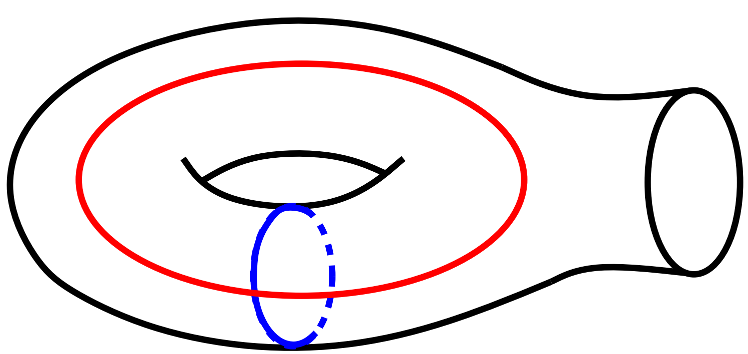
Extension of PTAS for Euclidean TSP

Uses Mitchell's guillotine rectangular subdivisions

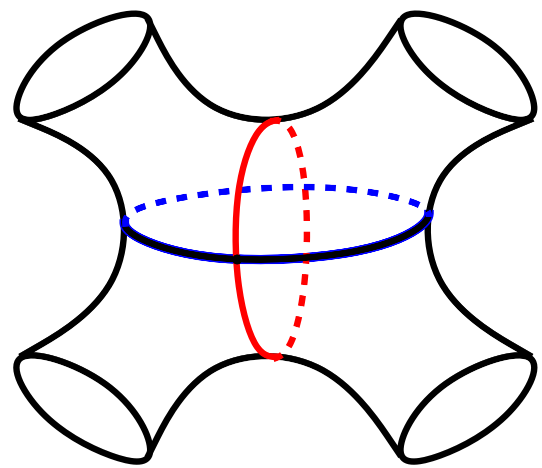
Previous work

Allen Hatcher.
Pants Decompositions of Surfaces.
arxiv.org/abs/math.GT/9906084

The *pants decomposition complex* of a given surface is simply connected—vertices are isotopy classes of pants decompositions, edges correspond to elementary moves



S-move



A-move

Previous work

Éric Colin de Verdière and Francis Lazarus.
Optimal Pants Decompositions and Shortest Homotopic Cycles on an Orientable Surface.
Graph Drawing, pp. 478–490, 2003 (+EuroCG'03)

Show how to *shorten* a given pants decomposition

Given a pants decomposition of a general combinatorial surface, they compute a *homotopic* pants decomposition in which each cycle is shortest in its homotopy class

Work in progress

NP-complete for general surfaces?

. . . I believe so!

NP-complete for the punctured plane?

. . . I don't know

David Eppstein recently obtained an $O(n \log n)$-time algorithm to compute an $O(1)$-approximation for the punctured plane using quadtrees [to appear in SODA 2007]

Thank you!